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On integration of multidimensional generalizations of classical C- and S-integrable nonlinear partial differential equations

A I Zenchuk

Institute of Problems of Chemical Physics, RAS Acad. Semenov av., 1 Chernogolovka, Moscow region 142432, Russia

E-mail: zenchuk@itp.ac.ru

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Abstract
We develop a new integration technique allowing one to construct a rich manifold of particular solutions to multidimensional generalizations of classical C- and S-integrable partial differential equations (PDEs). Generalizations of (1+1)-dimensional C-integrable and (2+1)-dimensional S-integrable N-wave equations are derived among examples. Examples of multidimensional second-order PDEs are represented as well.

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1. Introduction

After the original work [1], the integrability technique was intensively developing. At the present time, it covers a large class of nonlinear partial differential equations (PDEs) applicable in different branches of physics and mathematics. One should mention a few of the most popular integration methods, such as (a) the linearization by some direct substitution, for instance, by the Hopf–Cole substitution [2] and by its multidimensional generalization [3] (appropriate nonlinear PDEs are referred to as C-integrable PDEs [4–9]); (b) the method of characteristics [10] and its matrix generalization [11, 12] (Ch-integrable PDEs); (c) the inverse spectral transform method [1, 13, 14], the dressing method [15–19] and the Sato approach [20] (S-integrable PDEs). S-integrable nonlinear PDEs are most applicable in physics. We recall a few types of these equations: the soliton equations in (1+1)-dimensions, such as the Korteweg-de Vries (KdV) [1, 21] and the nonlinear Shrödinger (NLS) [22] equations; the soliton (2+1)-dimensional equations, such as the Kadomtsev–Petviashvili (KP) [23] and the Davey–Stewartson (DS) [24] equations; the self-dual-type PDEs having instanton solutions, like the self-dual Yang–Mills equation (SDYM); PDEs associated with commuting vector fields

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[25–31]. Nevertheless, the class of completely integrable nonlinear PDEs is very restrictive. Thus, extensions of the integrability technique on new types of nonlinear PDEs is an actual problem.

In this paper we suggest a new version of the dressing method allowing one to construct a rich manifold of particular solutions to a new class of nonlinear PDEs in any dimension. The novelty of this class of PDEs is that the integrability technique does not generate commuting flows to them in the usual sense, unlike all methods mentioned above, where any nonlinear PDE appears together with the commuting hierarchy of nonlinear PDEs. We show that our algorithm may provide arbitrary functions of $m-1$ independent variables in the solution space to m-dimensional PDEs, which suggests us to consider these PDEs as candidates for completely integrable PDEs. However, we do not represent rigorous justification of complete integrability.

To anticipate, we give simple examples of nonlinear PDEs for the matrix field V, derived in this paper.

(1) The system of first-order D-dimensional PDEs

$$\sum_{m=1}^{D}\left(V_{t_m}+VC^{(m)}V\right)B^{(m)}=0, \tag{1}$$

where $B^{(m)}$ and $C^{(m)}$ are some constant matrices. This is multidimensional generalization of the C-integrable (1+1)-dimensional nonlinear N-wave equation.

A simple example of this equation corresponds to $D=2, t_1=x, t_2=y, B_2^{(1)}=B_1^{(2)}=0$:

$$V=\begin{pmatrix} u & q\\ p & v\end{pmatrix}. \tag{2}$$

Then equation (1) reads

$$\begin{aligned}
&u_x+u^2C_1^{(1)}+pqC_2^{(1)}=0,\\
&p_x+p\left(uC_1^{(1)}+vC_2^{(1)}\right)=0,\\
&v_y+v^2C_2^{(2)}+pqC_1^{(2)}=0,\\
&q_y+q\left(uC_1^{(2)}+vC_2^{(2)}\right)=0,
\end{aligned} \tag{3}$$

which reduces to the Liouville equation

$$f_{xy}=C_2^{(1)}C_1^{(2)}\exp(2f), \tag{4}$$

if $C_1^{(1)}=C_2^{(2)}=0$ and $q=p=\mathrm{e}^f$. Here and below, $B_\alpha^{(m)}$ and $C_\alpha^{(m)}$ mean the αth diagonal elements of the diagonal matrices $B^{(m)}$ and $C^{(m)}$, respectively.

(2) The first-order D_1D_2-dimensional PDEs

$$\sum_{m_2=1}^{D_2}\sum_{m_1=1}^{D_1}L^{(m_1)}\left(V_{t_{m_1m_2}}+VC^{(m_1m_2)}V\right)R^{(m_2)}=0, \tag{5}$$

where $L^{(m_1)}, C^{(m_1m_2)}$ and $R^{(m_2)}$ are some constant matrices. The simple example corresponds to $D_1=D_2=2, L_\alpha^{(m_1)}=R_\beta^{(m_2)}=0$ if $\alpha\neq m_1$ and $\beta\neq m_2$, respectively, $t_{11}=x, t_{12}=y, t_{21}=z, t_{22}=t$. Let V be given by equation (2); then equation (5) yields

$$\begin{aligned}
&u_x+u^2C_1^{(11)}+pqC_2^{(11)}=0,\\
&p_z+p\left(uC_1^{(21)}+vC_2^{(21)}\right)=0,\\
&v_t+v^2C_2^{(22)}+pqC_1^{(22)}=0,\\
&q_y+q\left(uC_1^{(12)}+vC_2^{(12)}\right)=0,
\end{aligned} \tag{6}$$

which reduces to the following four-dimensional generalization of the Liouville equation (4):

$$g_{xy} = C_2^{(11)} C_1^{(12)} \,\mathrm{e}^{f+g}, \qquad f_{zt} = C_1^{(22)} C_2^{(21)} \,\mathrm{e}^{f+g} \tag{7}$$

if $C_1^{(11)} = C_2^{(22)} = C_2^{(12)} = C_1^{(21)} = 0$, $p = \exp f$, $q = \exp g$.

In particular, equation (5) reduces to the following $D_0(D_0-1)/2$-dimensional PDE, $D_1 = D_2 = D_0$:

$$\begin{aligned}
&\sum_{m_2=1}^{D_0} \sum_{\substack{m_1=1\\ m_2>m_1}}^{D_0} \Big[\mathrm{i}\left(L^{(m_1)} V_{\tau_{m_1m_2}} L^{(m_2)} - L^{(m_2)} V_{\tau_{m_1m_2}} L^{(m_1)}\right)\\
&\qquad\qquad + L^{(m_1)} V C^{(m_1m_2)} V L^{(m_2)} - L^{(m_2)} V C^{(m_1m_2)} V L^{(m_1)}\Big] = 0,\\
&\qquad\qquad t_{m_1m_2} = -\mathrm{i}\tau_{m_1m_2}, \qquad V = -V^{+}, \quad m_2 > m_1,
\end{aligned} \tag{8}$$

where $C^{(m_1m_2)} = -C^{(m_2m_1)}$, $C^{(m_1m_2)}$ and $L^{(m_1)}$ are the constant diagonal matrices and $^{+}$ means Hermitian conjugate. This equation has a physical meaning describing the interaction of $n_0(n_0-1)/2$ waves if V is the $n_0 \times n_0$ matrix. This is multidimensional generalization of the S-integrable (2+1)-dimensional nonlinear N-wave equation.

(3) The system of second-order D-dimensional PDEs

$$\sum_{m,n=1}^{D} \left(V_{t_nt_m} + (VC^{(n)}V)_{t_m} + VC^{(m)}V_{t_n} + VC^{(m)}VC^{(n)}V\right) B^{(mn)} = 0, \tag{9}$$

where $B^{(mn)}$ are the constant diagonal matrices. The scalar version of this equation reads

$$\begin{aligned}
&\sum_{m,n=1}^{D} V_{t_nt_m} B^{(mn)} + V \sum_{m=1}^{D} V_{t_m} \hat{C}^{(m)} + V^3 \hat{C} = 0,\\
&\hat{C}^{(m)} = 2\sum_{n=1}^{D} C^{(n)} B^{(mn)} + \sum_{m=1}^{D} C^{(m)} B^{(mn)}, \qquad \hat{C} = \sum_{m,n=1}^{D} C^{(m)} C^{(n)} B^{(mn)}.
\end{aligned} \tag{10}$$

Note that not all constant coefficients may be arbitrary in the above nonlinear PDEs. Constructing particular solutions, we will reveal some relations among coefficients.

Particular examples of nonlinear PDEs, such as equations (3), (6), (8), (10), will not be considered in this paper. Instead of this, we concentrate on the dressing algorithm allowing one to derive general equations, such as equations (1), (8), (9), and study the richness of available solution space for them.

The structure of this paper is as follows. We will derive generalization of the classical C-integrable first- and second-order nonlinear PDEs in section 2 with equations (1) and (9) as particular examples. Richness of the solution space to equation (1) will be discussed briefly and explicit particular solutions to this equation with $D = 3$ will be given. Generalization of the classical S-integrable nonlinear PDEs will be considered in section 3 with equations (5) and (8) as particular examples. Richness of the solution space to equation (5) will be discussed briefly and explicit particular solutions to this equation with $D_1 = D_2 = 2$ will be given. Conclusions will be represented in section 4.

2. Generalization of *C*-integrable nonlinear PDEs

2.1. Starting equations

Our algorithm is based on the following integral equation:

$$P(\mu) * \chi(\mu, \lambda; t) = W(\mu; t) * \chi(\mu, \lambda; t) + W(\lambda; t) \equiv W(\mu; t) * (\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda)), \tag{11}$$

where $P(\mu)$, $\chi(\mu, \lambda; t)$, $W(\lambda; t)$ are the $n_0 \times n_0$ matrix functions of arguments, t_i are the independent variables of nonlinear PDEs, $t = (t_1, \dots, t_D)$, $\mathrm{rank}(P) = n_0$, λ, μ, ν are the complex parameters. Here $*$ means the integral operator defined for any two functions $f(\mu)$ and $g(\mu)$ as follows:

$$f(\mu) * g(\mu) = \int f(\mu) g(\mu)\, \mathrm{d}\Omega_1(\mu), \tag{12}$$

and $\Omega_1(\mu)$ is some measure. We also introduce unit $\mathcal{I}_1(\lambda, \mu)$ and inverse $f^{-1}(\lambda, \mu)$ operators as follows:

$$\begin{aligned} &f(\lambda, \nu) * \mathcal{I}_1(\nu, \mu) = \mathcal{I}_1(\lambda, \nu) * f(\nu, \mu) = f(\lambda, \mu), \\ &f(\lambda, \nu) * f^{-1}(\nu, \mu) = f^{-1}(\lambda, \nu) * f(\nu, \mu) = \mathcal{I}_1(\lambda, \mu). \end{aligned} \tag{13}$$

We introduce parameters t_i through the function $\chi(\lambda, \mu; t)$, which is defined as a solution to the following system of linear equations:

$$\begin{aligned} \chi_{t_m}(\lambda, \mu; t) = {} & A^{(m)}(\lambda, \nu) * \chi(\nu, \mu; t) + \tilde{A}^{(m)}(\lambda) P(\nu) * \chi(\nu, \mu; t) + A^{(m)}(\lambda, \mu), \\ & m = 1, \dots, D, \end{aligned} \tag{14}$$

where $A^{(m)}(\lambda, \nu)$ and $\tilde{A}^{(m)}(\lambda)$ are the $n_0 \times n_0$ matrix functions of arguments. An important requirement to equation (11) is that it must be uniquely solvable with respect to $W(\lambda; t)$, i.e. the operator $*(\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda))$ must be invertable.

The matrices $A^{(m)}$ and $\tilde{A}^{(m)}$ may not be arbitrary. They have to provide compatibility of the system (14). We will show that there are two different methods which provide this compatibility. The first one (section 2.2) yields classical *C*-integrable nonlinear PDEs, linearizable by the multidimensional version of the Hopf–Cole transformation [2], while the second method (section 2.3) yields a new type of nonlinear PDEs whose complete integrability is not clarified yet. However, our algorithm supplies, at least, a rich manifold of particular solutions to these PDEs.

The following theorem is valid for both cases.

Theorem 2.1. *The matrix function $W(\lambda; t)$, obtained as a solution to the integral equation (11) with χ defined by equation (14), satisfies the following system of compatible linear equations:*

$$E^{(m)}(\lambda; t) := W_{t_m}(\lambda; t) + V^{(m)}(t) W(\lambda; t) + (W(\mu; t) - P(\mu)) * A^{(m)}(\mu, \lambda) = 0, \tag{15}$$

$$V^{(m)}(t) = (W(\mu; t) - P(\mu)) * \tilde{A}^{(m)}(\mu), \qquad m = 1, \dots, D. \tag{16}$$

Proof. To derive equation (15), we differentiate equation (11) with respect to t_m. Then, in view of equation (14), one obtains the following equation:

$$E^{(m)}(\mu; t) * (\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda)) = 0, \tag{17}$$

where $E^{(m)}$ is defined in equation (15). Since the operator $*(\chi(\mu, \lambda; t) + \mathcal{I}_1(\mu, \lambda))$ is invertable, equation (17) yields $E^{(m)}(\mu; t) = 0$, which coincides with equation (15). □

Remark. Following the classical integrability theory, we refer to equation (15) as the linear equation for the function $W(\lambda;t)$. However this is not correct because the functions $V^{(m)}(t)$ are defined in terms of $W(\lambda;t)$ by equation (16). Thus, strictly speaking, equation (15) is a nonlinear equation for $W(\lambda;t)$.

System (15) is an overdetermined system of compatible linear equations with potentials $V^{(m)}(t)$ in analogy with the classical integrability theory. In the classical theory, nonlinear PDEs for potentials $V^{(n)}$ may be obtained as compatibility conditions for the appropriate overdetermined linear system. However, this approach does not work in our case because of the last term in equation (15). Instead of this, we suggest a different method of derivation of nonlinear PDEs, see sections 2.2 and 2.3.

Now we analyze two methods that provide the compatibility of the system (14) and derive nonlinear PDEs associated with each of them.

2.2. *First method: classical C-integrable nonlinear PDEs*

In this subsection we write the compatibility condition of equation (14) as follows:

$$(A^{(m)}(\lambda,\nu)+\tilde{A}^{(m)}(\lambda)P(\nu))*\chi_{t_n}(\nu,\mu)=(A^{(n)}(\lambda,\nu)+\tilde{A}^{(n)}(\lambda)P(\nu))*\chi_{t_m}(\nu,\mu),\qquad \forall n,m. \tag{18}$$

Substituting equation (14) for derivatives of χ into equation (18) we obtain the following equation:

$$\begin{aligned}&(L^{(m)}*L^{(n)}-L^{(n)}*L^{(m)})*\chi+L^{(m)}*A^{(n)}-L^{(n)}*A^{(m)}=0,\\ &L^{(m)}(\lambda,\mu)=A^{(m)}(\lambda,\mu)+\tilde{A}^{(m)}(\lambda)P(\mu).\end{aligned} \tag{19}$$

Let equation (19) be satisfied for any function $\chi(\lambda,\mu;t)$ (which is a solution to the system (14)). Then equation (19) is equivalent to the following two equations relating the matrix functions $A^{(m)}$, $\tilde{A}^{(m)}$ and P:

$$L^{(m)}*A^{(n)}-L^{(n)}*A^{(m)}=0, \tag{20}$$

$$\begin{aligned}&L^{(m)}*L^{(n)}-L^{(n)}*L^{(m)}=0\quad \overset{\text{eq. (20)}}{\Rightarrow}\\ &(L^{(m)}(\lambda,\nu)*\tilde{A}^{(n)}(\nu)-L^{(n)}(\lambda,\nu)*\tilde{A}^{(m)}(\nu))P(\mu)=0.\end{aligned} \tag{21}$$

Since $\mathrm{rank}(P)=n_0$, equation (21) is equivalent to the following one:

$$\begin{aligned}&L^{(m)}*\tilde{A}^{(n)}-L^{(n)}*\tilde{A}^{(m)}=0\quad\Rightarrow\\ &A^{(m)}*\tilde{A}^{(n)}-A^{(n)}*\tilde{A}^{(m)}=\tilde{A}^{(n)}P*\tilde{A}^{(m)}-\tilde{A}^{(m)}P*\tilde{A}^{(n)},\qquad n,m=1,\dots,D.\end{aligned} \tag{22}$$

Equations (20) and (22) represent two constraints on the functions $A^{(m)}(\lambda,\mu)$ and $\tilde{A}^{(m)}(\lambda)$.

Now we have everything for derivation of nonlinear PDEs for the fields $V^{(m)}(t)$. For this purpose, let us consider the following combination of equations (15):

$$E^{(m)}(\lambda;t)*\tilde{A}^{(n)}(\lambda)-E^{(n)}(\lambda;t)*\tilde{A}^{(m)}(\lambda), \tag{23}$$

which yields, in view of equation (22),

$$V^{(n)}_{t_m}(t)-V^{(m)}_{t_n}(t)+V^{(m)}(t)V^{(n)}(t)-V^{(n)}(t)V^{(m)}(t)=0. \tag{24}$$

This equation is known to be linearizable by the Hopf–Cole transformation [2, 3]:

$$E^{(n)}_H:=\Psi_{t_n}(t)=\Psi(t)V^{(n)}(t), \tag{25}$$

where $\Psi(t)$ is an arbitrary $n_0 \times n_0$ matrix function of all variables t_i. The presence of an arbitrary function $\Psi(t)$ is associated with the fact that the system of nonlinear PDEs (24) is not complete. One needs one more equation relating $V^{(n)}, n = 1, \dots, D$.

To derive this additional equation we introduce either additional relations among $A^{(m)}$ and $\tilde{A}^{(m)}$ in our algorithm based on equation (11) or an additional linear PDE for Ψ in the classical algorithm based on equation (25) [3].

For instance, let

$$\sum_{m=1}^{D} A^{(m)}(\lambda,\nu) * \tilde{A}^{(n)}(\nu) B^{(m)} = -\sum_{m=1}^{D} \tilde{A}^{(m)}(\lambda) P(\nu) * \tilde{A}^{(n)}(\nu) B^{(m)} \tag{26}$$

and/or

$$\sum_{m=1}^{D} \Psi_{t_m} B^{(m)} = 0, \tag{27}$$

where $B^{(m)}$ are the $n_0 \times n_0$ arbitrary constant matrices. Then both the combination of equations (15), $\sum_{m=1}^{D} (E^{(n)}) B^{(m)}$, and the appropriate combination of equations (25), $\Psi^{-1} \sum_{m=1}^{D} \left(E_H^{(n)}\right)_{t_m} B^{(m)}$, yield the same nonlinear equation for $V^{(n)}$:

$$\sum_{m=1}^{D} \left(V_{t_m}^{(n)} + V^{(n)} V^{(m)}\right) B^{(m)} = 0, \qquad n = 1, \dots, D. \tag{28}$$

This N-wave-type equation is supplemented by constraints (24) [3].

In particular, introducing reduction $V^{(m)}(t) = V(t) C^{(m)}$ (where $C^{(m)}$ are the $n_0 \times n_0$ constant matrices) we reduce the system (24) into the following one:

$$V_{t_m} C^{(n)} - V_{t_n} C^{(m)} + V C^{(m)} V C^{(n)} - V C^{(n)} V C^{(m)} = 0, \qquad \forall n, m, \tag{29}$$

which is a (1+1)-dimensional hierarchy of commuting C-integrable N-wave equations.

Higher order nonlinear PDEs may be obtained in a similar way introducing appropriate equations instead of equation (26) and/or equation (27).

C-integrable PDEs will not be considered in this paper.

2.3. Second method: a new class of nonlinear PDEs

In this section we represent another way to provide the compatibility of equation (14). As a result we obtain a new class of nonlinear PDEs together with the rich manifold of particular solutions. In particular, there are solutions in the form of rational functions of exponents.

Let us use the following representation of $A^{(m)}(\lambda,\mu)$ and $\tilde{A}^{(m)}(\lambda)$:

$$A^{(m)}(\lambda,\mu) = \alpha^{(m)}(\lambda,\nu) \star \beta^{(m)}(\nu,\mu), \qquad \tilde{A}^{(m)}(\lambda) = \alpha^{(m)}(\lambda,\nu) \star \tilde{\beta}^{(m)}(\nu), \tag{30}$$

where $\alpha^{(m)}(\lambda,\nu)$, $\beta^{(m)}(\nu,\mu)$ and $\tilde{\beta}^{(m)}(\nu)$ are the $n_0 \times n_0$ matrix functions of arguments. Here, we introduce one more integral operator $\star$ defined for any two functions $f(\mu)$ and $g(\mu)$ as follows:

$$f(\mu) \star g(\mu) \equiv \int f(\mu) g(\mu)\, \mathrm{d}\Omega_2(\mu), \tag{31}$$

where $\Omega_2(\mu)$ is some measure. We also introduce unit $\mathcal{I}_2(\lambda,\mu)$ and inverse $f^{-1}(\lambda,\mu)$ operators as follows:

$$\begin{aligned} &f(\lambda,\nu) \star \mathcal{I}_2(\nu,\mu) = \mathcal{I}_2(\lambda,\nu) \star f(\nu,\mu) = f(\lambda,\mu), \\ &f(\lambda,\nu) \star f^{-1}(\nu,\mu) = f^{-1}(\lambda,\nu) \star f(\nu,\mu) = \mathcal{I}_2(\lambda,\mu). \end{aligned} \tag{32}$$

Now let us write equation (14) using representation (30) in the following form:

$$\chi_{t_m}(\lambda, \mu; t) = \alpha^{(m)}(\lambda, \nu) \star \xi^{(m)}(\nu, \mu; t). \tag{33}$$

Here

$$\xi^{(m)}(\lambda, \mu; t) = (\beta^{(m)}(\lambda, \nu) + \tilde{\beta}^{(m)}(\lambda) P(\nu)) * \chi(\nu, \mu; t) + \beta^{(m)}(\lambda, \mu), \qquad m = 1, \ldots, D. \tag{34}$$

It is obvious that the compatibility condition of the system (14) is equivalent to the compatibility condition of the system (33) which reads

$$\alpha^{(m)}(\lambda, \nu) \star \xi^{(m)}_{t_n}(\nu, \mu; t) = \alpha^{(n)}(\lambda, \nu) \star \xi^{(n)}_{t_m}(\nu, \mu; t), \qquad \forall n, m, \tag{35}$$

instead of equation (18). To satisfy this condition we assume the following relations between $\xi^{(m)}$ and $\xi^{(1)}$:

$$\xi^{(m)}(\lambda, \mu; t) = \eta^{(m)}(\lambda, \nu) \star \xi^{(1)}(\nu, \mu; t), \quad m > 1, \qquad \eta^{(1)}(\lambda, \mu) = \mathcal{I}_2(\lambda, \mu), \tag{36}$$

where $\eta^{(m)}(\lambda, \mu)$ are some $n_0 \times n_0$ matrix functions, which will be specified below. We also define t-dependence of $\xi^{(1)}$ as follows:

$$\xi^{(1)}_{t_m}(\lambda, \mu; t) = T^{(m)}(\lambda, \nu) \star \xi^{(1)}(\nu, \mu; t), \tag{37}$$

where $T^{(m)}(\lambda, \nu)$ are some $n_0 \times n_0$ matrix functions. Substituting equations (36) and (37) into equation (35) we obtain the following representation for $\alpha^{(m)}$, $m > 1$:

$$\alpha^{(m)}(\lambda, \mu) = \alpha^{(1)}(\lambda, \nu) \star T^{(m)}(\nu, \tilde{\nu}) \star (T^{(1)})^{-1}(\tilde{\nu}, \tilde{\mu}) \star (\eta^{(m)})^{-1}(\tilde{\mu}, \mu). \tag{38}$$

In turn, the compatibility condition of equation (37) requires

$$T^{(m)} \star T^{(n)} - T^{(n)} \star T^{(m)} = 0, \quad \Rightarrow \tag{39}$$

$$T^{(m)}(\lambda, \mu) = (T(\lambda, \nu)\tau^{(m)}(\nu)) \star T^{-1}(\nu, \mu), \qquad [\tau^{(m)}(\nu), \tau^{(n)}(\nu)] = 0, \quad \forall n, m, \tag{40}$$

where $T(\lambda, \mu)$ and $\tau^{(m)}(\mu)$ are some $n_0 \times n_0$ matrix functions of arguments. Thus, compatibility condition of the system (33) generates equations (36)–(40).

Now, integrating equation (37) with $m = 1$ we obtain

$$\xi^{(1)}(\lambda, \mu; t) = \left(T(\lambda, \nu)\, \mathrm{e}^{\sum_{i=1}^{D} \tau^{(i)}(\nu) t_i}\right) \star T^{-1}(\nu, \tilde{\nu}) \star \xi_0(\tilde{\nu}, \mu). \tag{41}$$

Finally, integrating equation (33) with $m = 1$, we derive the following explicit formula for $\chi(\lambda, \mu; t)$:

$$\chi(\lambda, \mu; t) = \alpha^{(1)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star \xi^{(1)}(\tilde{\nu}, \mu; t) + \chi_0(\lambda, \mu), \tag{42}$$

where $\chi_0(\lambda, \mu)$ is the $n_0 \times n_0$ matrix integration constant.

It is convenient to rewrite expressions for $A^{(m)}$ and $\tilde{A}^{(m)}$ (see equations (30)) using equation (38) as follows:

$$\begin{aligned}
&A^{(m)}(\lambda, \mu) = \alpha^{(1)}(\lambda, \nu) \star \Gamma^{(m)}(\nu, \mu), \qquad \tilde{A}^{(m)}(\lambda) = \alpha^{(1)}(\lambda, \nu) \star \tilde{\Gamma}^{(m)}(\nu), \\
&\Gamma^{(m)}(\lambda, \mu) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star (\eta^{(m)})^{-1}(\tilde{\mu}, \tilde{\lambda}) \star \beta^{(m)}(\tilde{\lambda}, \mu), \\
&\tilde{\Gamma}^{(m)}(\lambda) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star (\eta^{(m)})^{-1}(\tilde{\nu}, \tilde{\mu}) \star \tilde{\beta}^{(m)}(\tilde{\mu}).
\end{aligned} \tag{43}$$

Since $\eta^{(1)}(\lambda, \mu) = \mathcal{I}_2(\lambda, \mu)$, one has $\Gamma^{(1)} = \beta^{(1)}$ and $\tilde{\Gamma}^{(1)} = \tilde{\beta}^{(1)}$. Representations (43) will be used in the rest of section 2.3.

2.3.1. Internal constraints for $\Gamma^{(m)}$, $\tilde\Gamma^{(m)}$, T and $\tau^{(m)}$. Remember that definition of $\xi^{(m)}$ in terms of χ, see equation (34), must be consistent with equation (42). This requirement generates a set of constraints on $\Gamma^{(m)}$, $\tilde\Gamma^{(m)}$, T and $\tau^{(m)}$. To derive these constraints, we apply the operator $(\beta^{(m)}+\tilde\beta^{(m)}P)*$ to equation (42) from the left. One obtains

$$\xi^{(m)}-\beta^{(m)}=(\beta^{(m)}+\tilde\beta^{(m)}P)*\alpha^{(1)}\star(T^{(1)})^{-1}\star\xi^{(1)}+(\beta^{(m)}+\tilde\beta^{(m)}P)*\chi_0. \tag{44}$$

Substituting equation (36) for $\xi^{(m)}$ one has to obtain an identity valid for any $\xi^{(1)}$. This requirement yields, first of all, the following equation relating $\beta^{(m)}$, $\tilde\beta^{(m)}$ and χ_0 (the first constraint on the functions $\Gamma^{(m)}$ and $\tilde\Gamma^{(m)}$):

$$(\beta^{(m)}+\tilde\beta^{(m)}P)*\chi_0+\beta^{(m)}=0 \quad\Rightarrow \tag{45}$$

$$(\Gamma^{(m)}(\lambda,\nu)+\tilde\Gamma^{(m)}(\lambda)P(\nu))*\chi_0(\nu,\mu)+\Gamma^{(m)}(\nu,\mu)=0, \tag{46}$$

and the following definition of $\eta^{(m)}$:

$$\eta^{(m)}=(\beta^{(m)}+\tilde\beta^{(m)}P)*\alpha^{(1)}\star(T^{(1)})^{-1}, \qquad m=1,\ldots,D. \tag{47}$$

Finally, equation (47) written in terms of $\Gamma^{(m)}$ and $\tilde\Gamma^{(m)}$ yields

$$\begin{aligned}T^{(m)}(\lambda,\mu)={}&T^{(1)}(\lambda,\nu)*(\Gamma^{(m)}(\nu,\tilde\nu)+\tilde\Gamma^{(m)}(\nu)P(\tilde\nu))*\alpha^{(1)}(\tilde\nu,\tilde\mu)\star(T^{(1)})^{-1}(\tilde\mu,\mu),\\ &m=1,\ldots,D,\end{aligned} \tag{48}$$

where $T^{(m)}$ are represented by equation (40) in terms of T and $\tau^{(m)}$. This is the second constraint imposed on the matrix functions $\Gamma^{(m)}$, $\tilde\Gamma^{(m)}$, T and $\tau^{(m)}$.

Constraints obtained in this section are produced, generally speaking, by the system (14) and its compatibility condition. For this reason we refer to them as the internal constraints. In contrast, the external constraints will be introduced 'by hand' in order to derive the nonlinear PDEs, see theorems 2.2 and 2.3.

2.3.2. First-order nonlinear PDEs for the functions $V^{(m)}(t)$, $m=1,\ldots,D$.

Theorem 2.2. *In addition to equations (11), (14), we impose the following external constraint for the matrix functions $A^{(m)}(\lambda,\mu)$ and $\tilde A^{(m)}(\lambda)$:*

$$\sum_{m=1}^{D}A^{(m)}(\lambda,\nu)*\tilde A^{(n)}(\nu)B^{(mn)}=\sum_{m=1}^{D}\tilde A^{(m)}(\lambda)P^{(mn)}, \qquad n=1,\ldots,D, \tag{49}$$

where $B^{(mn)}$ and $P^{(mn)}$ are some $n_0\times n_0$ constant matrices. Then the $n_0\times n_0$ matrix functions $V^{(m)}(t)$, $m=1,\ldots,D$, are solutions to the following system of nonlinear PDEs:

$$\begin{aligned}&\sum_{m=1}^{D}\big[\big(V^{(n)}_{t_m}+V^{(m)}V^{(n)}\big)B^{(mn)}+V^{(m)}(\mathcal{A}^{(n)}B^{(mn)}+P^{(mn)})\big]=0, \qquad n=1,\ldots,D,\\ &\mathcal{A}^{(n)}=P(\lambda)*\tilde A^{(n)}(\lambda).\end{aligned} \tag{50}$$

Proof. Applying operator $*\tilde A^{(n)}$ to equation (15) from the right one obtains the following equation:

$$E^{(mn)}(t):=V^{(n)}_{t_m}(t)+V^{(m)}(t)(V^{(n)}(t)+\mathcal{A}^{(n)})+U^{(mn)}(t)=0, \tag{51}$$

which introduces a new set of fields $U^{(mn)}$:

$$U^{(mn)}(t)=(W(\mu;t)-P(\mu))*A^{(m)}(\mu,\lambda)*\tilde A^{(n)}(\lambda). \tag{52}$$

Due to relation (49), we may eliminate these fields using a proper combination of equations (51). Namely, the combination $\sum_{m=1}^{D}E^{(mn)}(t)B^{(mn)}$ results in the system (50). □

Reduction 1. Let

$$P^{(mn)} = -\mathcal{A}^n B^{(mn)}. \tag{53}$$

Then equation (50) reduces as follows:

$$\sum_{m=1}^{D} \left(V^{(n)}_{t_m} + V^{(m)} V^{(n)}\right) B^{(mn)} = 0. \tag{54}$$

Note that equation (54) coincides with the linearizable equation (28) if $B^{(mn)} = B^{(m)}$. However, equation (28) is supplemented by constraints (24), which are not valid for equation (54) in general. Constraint (49) reads in this case

$$\sum_{m=1}^{D} (A^{(m)}(\lambda,\nu) + \tilde{A}^{(m)}(\lambda) P(\nu)) * \tilde{A}^{(n)}(\nu) B^{(mn)} = 0, \qquad n = 1,\ldots,D. \tag{55}$$

Reduction 2. Let

$$\tilde{\Gamma}^{(m)}(\lambda) = \tilde{\Gamma}(\lambda) C^{(m)}, \qquad C^{(1)} \equiv I_{n_0}, \qquad C^{(n)} B^{(mn)} = B^{(m)}, \qquad m, n = 1,\ldots,D, \tag{56}$$

in addition to reduction (53). Here $C^{(m)}$ and $B^{(m)}$ are some $n_0 \times n_0$ constant matrices and $\tilde{\Gamma}(\lambda)$ is the $n_0 \times n_0$ matrix function. As a consequence, we obtain

$$\begin{aligned} &\tilde{A}^{(m)}(\lambda) = \tilde{A}(\lambda) C^{(m)}, && \tilde{A}(\lambda) = \alpha^{(1)}(\lambda,\mu) \star \tilde{\Gamma}(\mu), \\ &V^{(m)}(t) = V(t) C^{(m)}, && \tilde{\Gamma}(\lambda) = \tilde{\Gamma}^{(1)}(\lambda). \end{aligned} \tag{57}$$

Then the system of D (equation (54)) reduces to the following single PDE:

$$\sum_{m=1}^{D} (V_{t_m} + V C^{(m)} V) B^{(m)} = 0. \tag{58}$$

This equation is written in the introduction, see equation (1), and may be referred to as a multidimensional generalization of the (1+1)-dimensional C-integrable N-wave equation (29). Reduction (56) changes internal constraints (46) and (48) as follows:

$$(\Gamma^{(m)}(\lambda,\nu) + \tilde{\Gamma}(\lambda) C^{(m)} P(\nu)) * \chi_0(\nu,\mu) + \Gamma^{(m)}(\lambda,\mu) = 0, \tag{59}$$

$$\begin{aligned} T^{(m)}(\lambda,\mu) = T^{(1)}(\lambda,\nu) \star (\Gamma^{(m)}(\nu,\tilde{\nu}) + \tilde{\Gamma} C^{(m)}(\nu) P(\tilde{\nu})) * \alpha^{(1)}(\tilde{\nu},\tilde{\mu}) \star (T^{(1)})^{-1}(\tilde{\mu},\mu), \\ m = 1,\ldots,D. \end{aligned} \tag{60}$$

In turn, external constraint (55) reduces to the following single equation:

$$\sum_{m=1}^{D} (A^{(m)}(\lambda,\nu) + \tilde{A}(\lambda) C^{(m)} P(\nu)) * \tilde{A}(\nu) B^{(m)} = 0. \tag{61}$$

2.3.3. Second-order nonlinear PDEs for the functions $V^{(m)}$, $m = 1,\ldots,D$.

Theorem 2.3. *In addition to equations (11), (14), we impose the following external constraint for the matrix functions $A^{(m)}(\lambda,\mu)$ and $\tilde{A}^{(m)}(\lambda)$ (instead of constraint (49)):*

$$\begin{aligned} &\sum_{m,n=1}^{D} A^{(m)}(\lambda,\nu) * A^{(n)}(\nu,\mu) * \tilde{A}^{(l)}(\mu) B^{(mnl)} \\ &\quad = \sum_{m,n}^{D} A^{(m)}(\lambda,\mu) * \tilde{A}^{(n)}(\mu) P^{(mnl)} + \sum_{m=1}^{D} \tilde{A}^{(m)}(\lambda) P^{(ml)}, \end{aligned} \tag{62}$$

where $B^{(mnl)}$, $P^{(mnl)}$ and $P^{(ml)}$ are some constant $n_0 \times n_0$ matrices. Then the $n_0 \times n_0$ matrix functions $V^{(m)}(t)$ are solutions to the following system of nonlinear PDEs:

$$\sum_{m,n=1}^{D} \left[V^{(l)}_{t_n t_m} + (V^{(n)}V^{(l)})_{t_m} + V^{(m)}V^{(l)}_{t_n} + V^{(m)}V^{(n)}V^{(l)}\right]B^{(mnl)} + \sum_{m,n=1}^{D} \left(V^{(n)}_{t_m} + V^{(m)}V^{(n)}\right)\mathcal{A}_1^{(mnl)} + \sum_{m=1}^{D} V^{(m)}\mathcal{A}_2^{(ml)} = 0, \qquad l = 1, \ldots, D, \tag{63}$$

$$\mathcal{A}_1^{(mnl)} = \mathcal{A}^{(l)}B^{(mnl)} + P^{(mnl)}, \qquad \mathcal{A}_2^{(ml)} = \sum_{n=1}^{D} \mathcal{A}^{(n)}P^{(mnl)} - P^{(ml)}.$$

Proof. Applying the operator $*\tilde{A}^{(n)}$ to equation (15) from the right one obtains equation (51), which introduces a new set of fields $U^{(mn)}$ defined by equation (52). However, we may not eliminate these fields from the system (51) because constraint (49) is not valid in this theorem. Instead of this, we derive nonlinear PDEs for these fields as follows. Applying the operator $*A^{(n)} * \tilde{A}^{(l)}$ to equation (15) from the right one obtains

$$E^{(mnl)}(t) := U^{(nl)}_{t_m}(t) + V^{(m)}(t)U^{(nl)}(t) + U^{(mnl)}(t) = 0, \tag{64}$$

where one more set of matrix fields appears:

$$U^{(mnl)}(t) = (W(\lambda; t) - P(\lambda)) * A^{(m)}(\lambda, \nu) * A^{(n)}(\nu, \mu) * \tilde{A}^{(l)}(\mu). \tag{65}$$

Due to constraint (62), these fields may be eliminated in a proper combination of equations (64), namely $\sum_{m,n=1}^{D} E^{(mnl)}B^{(mnl)}$. Then, substituting $U^{(mn)}$ from equation (51) one ends up with equation (63). □

Emphasize that nonlinear equations (50) and (63) do not represent commuting flows since we assume different external constraints (49) and (62) for the matrix functions $A^{(m)}(\lambda, \mu)$ and $\tilde{A}^{(m)}(\lambda)$ deriving these equations. These two constraints are not compatible in general.

Reduction 1. Let

$$P^{(mnl)} = -\mathcal{A}^{(l)}B^{(mnl)}, \qquad P^{(ml)} = \sum_{n=1}^{D} \mathcal{A}^{(n)}P^{(mnl)}. \tag{66}$$

Then equation (63) reads

$$\sum_{m,n=1}^{D} \left[V^{(l)}_{t_n t_m} + (V^{(n)}V^{(l)})_{t_m} + V^{(m)}V^{(l)}_{t_n} + V^{(m)}V^{(n)}V^{(l)}\right]B^{(mnl)} = 0, \qquad l = 1, \ldots, D. \tag{67}$$

Constraint (62) reduces to the following one:

$$\sum_{m,n=1}^{D} (A^{(m)}(\lambda, \nu) * A^{(n)}(\nu, \mu) + A^{(m)}(\lambda, \nu) * \tilde{A}^{(n)}(\nu)P(\mu) + \tilde{A}^{(m)}(\lambda)P(\nu) * \tilde{A}^{(n)}(\nu)P(\mu)) * \tilde{A}^{(l)}(\mu)B^{(mnl)} = 0. \tag{68}$$

Reduction 2. Along with reduction (66) we consider reduction (56), (57) with

$$C^{(l)} B^{(mnl)} = B^{(mn)}, \tag{69}$$

where $B^{(mn)}$ are some constant matrices. System (67) reduces to the following single PDE:

$$\sum_{m,n=1}^{D} \left[V_{t_n t_m} + (VC^{(n)}V)_{t_m} + VC^{(m)}V_{t_n} + VC^{(m)}VC^{(n)}V \right] B^{(mn)} = 0. \tag{70}$$

This equation is written in the introduction, see equation (9). Internal constraints (59) and (60) remain valid for this case as well. External constraint (68) reduces to the following single equation:

$$\sum_{m,n=1}^{D} (A^{(m)}(\lambda,\nu) * A^{(n)}(\nu,\mu) + A^{(m)}(\lambda,\nu) * \tilde{A}(\nu) C^{(n)} P(\mu)$$
$$+ \tilde{A}(\lambda) C^{(m)} P(\nu) * \tilde{A}(\nu) C^{(n)} P(\mu)) * \tilde{A}(\mu) B^{(mn)} = 0. \tag{71}$$

2.3.4. Solutions to the first-order nonlinear PDE (58). The problem of richness of the available solution space will be considered for the nonlinear PDE (58). We show that solution space to this equation may be full provided that all constraints (59)–(61) may be resolved keeping proper arbitrariness of the functions $\tau^{(m)}(\nu)$. Examples of particular solutions will be considered as well.

On the dimensionality of the available solution space. We estimate the dimensionality of solution space for small χ. In this case formula (11) yields $W(\lambda; t) \approx P(\nu) * \chi(\nu, \lambda; t)$ and formula (16) gives us

$$V(t) \approx (P(\nu) * \chi(\nu, \lambda; t) - P(\lambda)) * \tilde{A}(\lambda). \tag{72}$$

By construction, if all $\tau^{(m)}(\nu)$ $(m = 1, \ldots, D)$ are the arbitrary functions, this expression preserves the following arbitrary $n_0 \times n_0$ matrix function of all D variables:

$$\begin{aligned} F(t) &= P * \alpha^{(1)} \star (T^{(1)})^{-1} \star \xi^{(1)} * \tilde{A} \equiv \int g(\nu)\, \mathrm{e}^{\sum_{i=1}^{D} \tau^{(i)}(\nu) t_i} g_2(\nu)\, \mathrm{d}\Omega_2(\nu), \\ g_1(\nu) &= P(\tilde{\nu}) * \alpha^{(1)}(\tilde{\nu}, \mu) \star (T^{(1)})^{-1}(\mu, \lambda) \star T(\lambda, \nu), \\ g_2(\nu) &= T^{-1}(\nu, \tilde{\nu}) \star \xi_0(\tilde{\nu}, \mu) * \tilde{A}(\mu). \end{aligned} \tag{73}$$

However, dimensionality of this function reduces due to the presence of constraints (59)–(61) which impose relations among $\tau^{(m)}(\nu)$. An important question is whether the dimensionality of function (73) may be equal to $D - 1$, which is necessary for fullness of the solution space. At first glance, we may expect the positive answer. In fact, equation (59) may be satisfied using special structures of $\Gamma^{(m)}$, P and χ_0, as it is done in the example below, see equations (88). Next, equation (60) relates $\tau^{(m)}$ with $\Gamma^{(m)}$ and $\tilde{\Gamma}^{(m)}$ which, in general, keeps arbitrariness of all $\tau^{(m)}(\nu)$ and consequently does not restrict dimensionality of the above written arbitrary function. Finally, equation (61) must be considered as a single relation among $\tau^{(m)}(\nu)$, $m = 1, \ldots,$ reducing the dimensionality of function (73) from D to $D - 1$, which means the full dimensionality of the solution space. Thus, we may expect new completely integrable nonlinear PDEs in the derived class of equations.

We have outlined a rough analysis of the solution space dimensionality. More detailed analysis must be carried out for particular equations and remains beyond the scope of this paper.

Construction of explicit solutions. Now we derive a family of particular solutions in the form of rational functions of exponents. Solitons and kinks are the most famous representatives of this family. To derive such solutions, we take

$$\mathrm{d}\Omega_1(\lambda) = \sum_{i=1}^{M} \delta(\lambda - a_i)\,\mathrm{d}\lambda, \qquad \mathrm{d}\Omega_2(\lambda) = \sum_{i=1}^{N} \delta(\lambda - b_i)\,\mathrm{d}\lambda, \qquad T(\lambda,\mu) = \mathcal{I}_2(\lambda,\mu),$$
$$\mathcal{I}_1 \to I_{Mn_0}, \qquad \mathcal{I}_2 \to I_{Nn_0}. \tag{74}$$

Then all integral equations reduce to the algebraic ones. We use the notations

$$\hat W = \begin{bmatrix} W(a_1) & \cdots & W(a_M)\end{bmatrix}, \qquad \hat\chi = \begin{bmatrix} \chi(a_1,a_1) & \cdots & \chi(a_1,a_M)\\ \cdots & \cdots & \cdots \\ \chi(a_M,a_1) & \cdots & \chi(a_M,a_M)\end{bmatrix},$$
$$\hat\chi_0 = \begin{bmatrix} \chi_0(a_1,a_1) & \cdots & \chi_0(a_1,a_M)\\ \cdots & \cdots & \cdots \\ \chi_0(a_M,a_1) & \cdots & \chi_0(a_M,a_M)\end{bmatrix}, \qquad \hat\alpha^{(1)} = \begin{bmatrix} \alpha^{(1)}(a_1,b_1) & \cdots & \alpha^{(1)}(a_1,b_N)\\ \cdots & \cdots & \cdots \\ \alpha^{(1)}(a_M,b_1) & \cdots & \alpha^{(1)}(a_M,b_N)\end{bmatrix},$$
$$\hat\Gamma^{(m)} = \begin{bmatrix} \Gamma^{(m)}(b_1,a_1) & \cdots & \Gamma^{(m)}(b_1,a_M)\\ \cdots & \cdots & \cdots \\ \Gamma^{(m)}(b_N,a_1) & \cdots & \Gamma^{(m)}(b_N,a_M)\end{bmatrix}, \qquad \hat{\tilde\Gamma} = \begin{bmatrix} \tilde\Gamma(b_1)\\ \cdots \\ \tilde\Gamma(b_N)\end{bmatrix},$$
$$\hat\xi_0 = \begin{bmatrix} \xi_0(b_1,a_1) & \cdots & \xi_0(b_1,a_M)\\ \cdots & \cdots & \cdots \\ \xi_0(b_N,a_1) & \cdots & \xi_0(b_N,a_M)\end{bmatrix}, \qquad \hat P = [P(a_1) \quad \cdots \quad P(a_M)],$$
$$\hat\tau^{m)} = \mathrm{diag}(\tau^{(m)}(b_1), \cdots, \tau^{(m)}(b_N)). \tag{75}$$

Solution V is given by equation (16) together with reduction (56), (57) as follows:

$$V = (\hat W - \hat P)\hat{\tilde A} \tag{76}$$

where $\hat W$ is a solution to equation (11):

$$\hat W = \hat P \hat\chi \left(\hat\chi + I_{Mn_0}\right)^{-1}. \tag{77}$$

Substituting equation (77) into equation (76) we obtain

$$V = \hat P\left(\hat\chi\left(\hat\chi + I_{Mn_0}\right)^{-1} - I_{Mn_0}\right)\hat\alpha^{(1)}\hat{\tilde\Gamma}. \tag{78}$$

Since χ is given by equation (42), one has

$$\hat\chi = \hat\alpha^{(1)}(\hat\tau^{(1)})^{-1}\,\mathrm{e}^{\sum_{i=1}^{D}\hat\tau^{(i)}t_i}\hat\xi_0 + \hat\chi_0, \tag{79}$$

where we substitute equation (41) for $\xi^{(1)}$.

The matrices $\hat\Gamma^{(m)}$, $\hat{\tilde\Gamma}$ and $\hat\tau^{(m)}$ must satisfy constraints (59)–(61), which read in our case, $m = 1, \ldots, D$,

$$(\hat\Gamma^{(m)} + \hat{\tilde\Gamma}C^{(m)}\hat P)\chi_0 + \hat\Gamma^{(m)} = 0, \tag{80}$$
$$\hat\tau^{(m)} = \hat\tau^{(1)}(\hat\Gamma^{(m)} + \hat{\tilde\Gamma}C^{(m)}\hat P)\hat\alpha^{(1)}(\hat\tau^{(1)})^{-1}, \tag{81}$$
$$\sum_{m=1}^{D}\hat\alpha^{(1)}(\hat\Gamma^{(m)} + \hat{\tilde\Gamma}C^{(m)}\hat P)\hat\alpha^{(1)}\hat{\tilde\Gamma}B^{(m)} = 0. \tag{82}$$

Analysis of equation (80) points on two different types of solutions to them. The first type is associated with $\det\left(\hat\chi_0 + I_{Mn_0}\right) \neq 0$. Then, equation (80) may be solved for $\Gamma^{(m)}$ and one can show that multidimensional PDE (58) may be split into a set of independent compatible ordinary differential equations (ODEs). We will not consider this case. The second type is associated with $\det\left(\hat\chi_0 + I_{Mn_0}\right) = 0$ and leads to truly multidimensional solutions to equation (58). Namely this case is considered hereafter.

Looking for the particular solutions to equation (80) we decompose it into two equations:

$$\hat P\hat\chi_0 = 0, \qquad \hat\Gamma^{(m)}\left(\hat\chi_0 + I_{Mn_0}\right) = 0, \tag{83}$$

which means that the rows of $\hat P$ and $\hat\Gamma^{(m)}$ are orthogonal to the columns of $\hat\chi_0$ and $\hat\chi_0 + I_{Mn_0}$, respectively.

Note that equation (81) with $m = 1$ reads

$$\hat\tau^{(1)} = (\hat\Gamma^{(1)} + \hat{\hat\Gamma}\hat P)\hat\alpha^{(1)}. \tag{84}$$

Since, $\hat\tau^{(1)}$ must be invertable, we require $M > N$. Then equation (82) may be simplified removing $\hat\alpha^{(1)}$ as a left factor in this equation:

$$\sum_{m=1}^{D}(\hat\Gamma^{(m)} + \hat{\hat\Gamma}C^{(m)}\hat P)\hat\alpha^{(1)}\hat{\hat\Gamma}B^{(m)} = 0. \tag{85}$$

Simple example of solution. We consider the three-dimensional nonlinear PDE (58), i.e. $D = 3$:

$$\sum_{m=1}^{3}(V_{t_m} + VC^{(m)}V)B^{(m)} = 0. \tag{86}$$

Let $n_0 = 2$, $M = 5$, $N = 2$,

$$B^{(1)} = C^{(1)} = I_2, \qquad B^{(i)} = \operatorname{diag}\left(b_1^{(i)}, b_2^{(i)}\right), \qquad C^{(i)} = \operatorname{diag}\left(c_1^{(i)}, c_2^{(i)}\right), \quad i = 2, 3. \tag{87}$$

In order to satisfy equations (81), (83) and (85) we take the following matrices $\hat\chi_0$, $\hat P$, $\hat\Gamma^{(m)}$, $\hat{\hat\Gamma}$:

$$\begin{aligned}
&\hat\chi_0 = \begin{bmatrix} -I_2 & Z_{2,8} \\ F_{2,2} & F_{2,8} \\ Z_{6,2} & Z_{6,8}\end{bmatrix}, \qquad F_{2,2} = Z, \qquad F_{2,8} = [J_4\ Z\ J_5\ Z],\\
&\hat\alpha^{(1)} = \begin{bmatrix} I_4 \\ I_4 \\ J_0\end{bmatrix}, \qquad \hat{\hat\xi}_0 = (\alpha^{(1)})^T\\
&\hat P = [Z\ Z\ Z\ J_1\ Z],\\
&\hat\Gamma^{(1)} = \begin{bmatrix} J_2 & Z & Z & Z & Z \\ Z & Z & Z & Z & Z\end{bmatrix}, \qquad \hat\Gamma^{(2)} = \hat\Gamma^{(3)} = Z_{4,10},\\
&\hat{\hat\Gamma} \equiv \hat{\hat\Gamma}^{(1)} = \begin{bmatrix} Z \\ J_3 \end{bmatrix}, \qquad \hat\tau^{(1)} = \operatorname{diag}(1, 2, 3, 4), \qquad \hat\tau^{(2)} = \operatorname{diag}\left(0, 0, 3c_2^{(2)}, 4c_1^{(2)}\right),\\
&\hat\tau^{(3)} = \operatorname{diag}\left(0, 0, 3c_2^{(3)}, 4c_1^{(3)}\right).
\end{aligned} \tag{88}$$

In addition, we obtain expressions for $c_i^{(3)}$, $i = 1, 2$:

$$c_i^{(3)} = -\frac{1 + b_i^{(2)}c_i^{(2)}}{b_i^{(3)}}, \qquad i = 1, 2. \tag{89}$$

Here $Z_{i,j}$ and Z are the $i \times j$ and 2×2 zero matrices, respectively:

$$J_0 = [I_2\ \ Z], \qquad J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, \qquad J_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2\end{bmatrix}, \qquad J_3 = \begin{bmatrix} 0 & 3 \\ 4 & 0\end{bmatrix}, \tag{90}$$

$$J_4 = \begin{bmatrix} \dfrac{8p_3 + p_2(6 - p_3) - 4p_4 - p_1(12 - p_4)}{2(6p_1 - 3p_2 - 2p_3 + p_4)} & 1 \\ \dfrac{(-12 + 3p_2 - p_4)(-2p_3 + p_4)}{12(6p_1 - 3p_2 - 2p_3 + p_4)} & 0 \end{bmatrix},$$
$$J_5 = \begin{bmatrix} 0 & 1 + \dfrac{4(3p_1 - p_3 - 6)}{12 - 3p_2 + p_4} \\ -1 + \dfrac{p_2}{4} - \dfrac{p_4}{12} & 0 \end{bmatrix}. \tag{91}$$

Diagonal elements of the matrices $B^{(i)}, i = 2, 3$, and $C^{(2)}$ remain arbitrary. Substituting equations (88) into equations (78), (79) one obtains V as a rational expression of exponents:

$$V(t) = \frac{1}{D}\begin{bmatrix} -4(e^{\eta_2} p_2 + p_4) & f_{12}\, e^{\eta_1} \\ f_{21}\, e^{\eta_2} & -3(e^{\eta_1} p_3 + p_4) \end{bmatrix},$$
$$D = e^{\eta_1+\eta_2} p_1 + e^{\eta_2} p_2 + e^{\eta_1} p_3 + p_4, \tag{92}$$
$$f_{12} = -3\frac{(-12 + 3p_2 - p_4)(p_2 p_3 - p_1 p_4)}{4(-12p_1 + 6p_2 + 4p_3 - 2p_4)}, \qquad f_{21} = 16\frac{12p_1 - 6p_2 - 4p_3 + 2p_4}{-12 + 3p_2 - p_4}.$$

Here

$$\eta_1 = 4t_1 + 4c_1^{(2)} t_2 - \frac{4}{b_1^{(3)}}\left(1 + b_1^{(2)} c_1^{(2)}\right)t_3,$$
$$\eta_1 = 3t_1 + 3c_2^{(2)} t_2 - \frac{3}{b_2^{(3)}}\left(1 + b_2^{(2)} c_2^{(2)}\right)t_3. \tag{93}$$

We see that all elements of V are kinks.

Relations (89) show that not all matrix coefficients in equation (86) are arbitrary. They are related by the following equation:

$$C^{(3)} B^{(3)} + C^{(2)} B^{(2)} + I_2 = 0. \tag{94}$$

Thus, we have constructed a particular solution to the three-dimensional nonlinear PDE (86) with the diagonal matrices $C^{(i)}$, $B^{(i)}$ related by equation (94).

3. Multidimensional generalization of *S*-integrable PDEs

3.1. Starting equations

Algorithm developed in this section is based on the same equation (11) with different function $\chi(t)$, which is defined by the following system of equations:

$$\chi_{t_m}(\lambda, \mu; t) = (A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda)P(\nu)) * \chi(\nu, \mu; t) - \chi(\lambda, \nu; t) * A^{(m)}(\nu, \mu),$$
$$m = 1, \ldots, D, \tag{95}$$

instead of the system (14). Here, again, $A^{(m)}(\lambda, \nu)$ and $\tilde{A}^{(m)}(\lambda)$ are the $n_0 \times n_0$ matrix functions of arguments.

The matrices $A^{(m)}$ and $\tilde{A}^{(m)}$ have to provide compatibility of the system (95). Similar to equation (14), there are two different methods that provide this compatibility. The first one yields the classical *S*-integrable nonlinear PDEs, section 3.2, while the second method yields a new type of nonlinear PDEs whose complete integrability is not clarified yet, section 3.3. However, our algorithm supplies, at least, a rich manifold of particular solutions to these PDEs.

3.2. First method: classical S-integrable (2+1)-dimensional N-wave equation

Consider the compatibility condition of equation (95) in the following form:

$$(A^{(m)}(\lambda,\nu)+\tilde A^{(m)}(\lambda)P(\nu))*\chi_{t_n}(\nu,\mu;t)-\chi_{t_n}(\lambda,\nu;t)*A^{(m)}(\nu,\mu)$$
$$=(A^{(n)}(\lambda,\nu)+\tilde A^{(n)}(\lambda)P(\nu))*\chi_{t_m}(\nu,\mu;t)-\chi_{t_m}(\lambda,\nu;t)*A^{(n)}(\nu,\mu),\qquad\forall n,m. \tag{96}$$

Substituting equation (95) for derivatives of χ we reduce equation (96) to the following one:

$$\begin{aligned}&(L^{(m)}*L^{(n)}-L^{(n)}*L^{(m)})*\chi-\chi*(A^{(m)}*A^{(n)}-A^{(n)}*A^{(m)})=0,\\&L^{(m)}(\lambda,\mu)=A^{(m)}(\lambda,\mu)+\tilde A^{(m)}(\lambda)P(\mu).\end{aligned}\tag{97}$$

Since equation (97) must be valid for any function $\chi(t)$ (which is a solution to the system (95)), it is equivalent to the following two equations relating the matrix functions $A^{(m)}$, $\tilde A^{(m)}$ and P:

$$\begin{aligned}&A^{(m)}(\lambda,\nu)*A^{(n)}(\nu,\mu)-A^{(n)}(\lambda,\nu)*A^{(m)}(\nu,\mu)=0,\\&L^{(m)}*L^{(n)}-L^{(n)}*L^{(m)}=0\quad\overset{\text{eq. (98)}}{\Rightarrow}\end{aligned}\tag{98}$$

$$L^{(m)}(\lambda,\nu)*\tilde A^{(n)}(\nu)P(\mu)-L^{(n)}(\lambda,\nu)*\tilde A^{(m)}(\nu)P(\mu)$$
$$=\tilde A^{(n)}(\lambda)P(\nu)*A^{(m)}(\nu,\mu)-\tilde A^{(m)}(\lambda)P(\nu)*A^{(n)}(\nu,\mu). \tag{99}$$

In order to satisfy equation (99) we require the following representation of $\tilde A^{(m)}(\lambda)$:

$$\tilde A^{(m)}(\lambda)=\tilde A(\lambda)B^{(m)},\qquad [B^{(m)},B^{(n)}]=0, \tag{100}$$

where $\tilde A(\lambda)$ and $B^{(m)}$ are the $n_0\times n_0$ matrix function and the constant matrix, respectively. Then equation (99) is equivalent to the following system:

$$L^{(m)}(\lambda,\nu)*\tilde A(\nu)B^{(n)}-L^{(n)}(\lambda,\nu)*\tilde A(\nu)B^{(m)}=0, \tag{101}$$

$$B^{(n)}P(\nu)*A^{(m)}(\nu,\mu)-B^{(m)}P(\nu)*A^{(n)}(\nu,\mu)=0. \tag{102}$$

Equations (98), (101), (102) represent three constraints for the matrices $A^{(m)}$ and $\tilde A$.

Theorem 3.1. *Let the matrix function $W(\lambda;t)$ be obtained as a solution to the integral equation (11) with χ defined by equation (95) supplemented with constraints (98), (101), (102). Then*

(1) the function $W(\lambda;t)$ satisfies the following system of compatible linear equations:

$$E^{(nm)}(\mu;t):=B^{(n)}\big(W_{t_m}(\mu;t)+V(t)B^{(m)}W(\mu;t)+W(\nu;t)*A^{(m)}(\nu,\mu)\big)$$
$$-B^{(m)}\big(W_{t_n}(\mu;t)+V(t)B^{(n)}W(\mu;t)+W(\nu;t)*A^{(n)}(\nu,\mu)\big)=0, \tag{103}$$

$$V(t)=(W(\nu;t)-P(\nu))*\tilde A(\nu),\qquad n,m=1,\dots,D; \tag{104}$$

(2) the matrix field $V(t)$, given by equation (104), satisfies the following S-integrable N-wave equation:

$$\sum_{\mathrm{perm}(n,m,l)}\big(B^{(n)}\big(V_{t_m}+VB^{(m)}V\big)-B^{(m)}\big(V_{t_n}+VB^{(n)}V\big)\big)B^{(l)}=0, \tag{105}$$

where perm(n,m,l) *means clockwise circle permutations.*

Proof.

(1) To derive equation (103), we differentiate equation (11) with respect to t_m. Then, in view of equation (95), one obtains the following equations:

$$\begin{aligned}&\mathcal{E}^{(m)}(\mu;t) := P(\lambda) * A^{(m)}(\lambda,\nu) * \chi(\nu,\mu;t) = E^{(m)}(\nu;t) * (\chi(\nu,\mu) + \mathcal{I}_1(\nu,\mu)),\\ &E^{(m)}(\mu;t) = W_{t_m}(\mu;t) + V^{(m)}(t)W(\mu;t) + W(\nu;t) * A^{(m)}(\nu,\mu).\end{aligned} \tag{106}$$

Due to constraint (102), lhs in equations (106) may be removed using the following combination of these equations:

$$B^{(n)}\mathcal{E}^{(m)} - B^{(m)}\mathcal{E}^{(n)} \quad \Rightarrow \tag{107}$$

$$(B^{(n)}E^{(m)}(\nu;t) - B^{(m)}E^{(n)}(\nu;t)) * (\chi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)) = 0. \tag{108}$$

Since the operator $*(\chi(\nu,\mu) + \mathcal{I}_1(\nu,\mu))$ is invertable, one obtains

$$\mathcal{E}^{(nm)} := B^{(n)}E^{(m)} - B^{(m)}E^{(n)} = 0, \tag{109}$$

which coincides with equation (103).

(2) In order to derive nonlinear PDE (105) we consider the following combination of equations (103):

$$\sum_{\mathrm{perm}(n,m,l)} E^{(nm)} * \tilde{A}B^{(l)}, \tag{110}$$

which yields equation (105) in view of constraint (101). □

S-integrable PDEs will not be considered in this paper.

3.3. Second method: a new class of nonlinear PDEs

We will use double indexes hereafter in this section, i.e.

$$\begin{aligned}&V^{(m)} \equiv V^{(m_1m_2)}, \qquad t_m \equiv t_{m_1m_2},\\ &\sum_{m=1}^{D} f^{(m)} \equiv \sum_{m_2=1}^{D_2}\sum_{m_1=1}^{D_1} f^{(m_1m_2)}, \quad \forall f, \qquad D \equiv (D_1, D_2),\end{aligned} \tag{111}$$

and notation $f^{(1)} \equiv f^{(11)}, \forall f$, for the sake of brevity. For instance, $\tau^{(1)} \equiv \tau^{(11)}$, $\eta^{(1)} \equiv \eta^{(11)}$ and so on. Similar to section 2.3, we will use representations (30) for the matrix functions $A^{(m)}(\lambda,\mu)$ and $\tilde{A}^{(m)}(\lambda)$ and write equation (95) in the following form:

$$\chi_{t_m}(\lambda,\mu;t) = \alpha^{(m)}(\lambda,\nu) \star \xi^{(m)}(\nu,\mu) - \bar{\xi}^{(m)}(\lambda,\nu) \star \beta^{(m)}(\nu,\mu), \tag{112}$$

$$\xi^{(m)}(\lambda,\mu) = (\beta^{(m)}(\lambda,\nu) + \tilde{\beta}^{(m)}(\lambda)P(\nu)) * \chi(\nu,\mu;t), \tag{113}$$

$$\bar{\xi}^{(m)}(\lambda,\mu) = \chi(\lambda,\nu) * \alpha^{(m)}(\nu,\mu), \qquad m_i = 1,\ldots,D_i, \quad i = 1,2. \tag{114}$$

Then the compatibility condition for the system (95) is equivalent to the compatibility condition for the system (112) which reads

$$\alpha^{(m)} \star \xi^{(m)}_{t_n} - \bar{\xi}^{(m)}_{t_n} \star \beta^{(m)} = \alpha^{(n)} \star \xi^{(n)}_{t_m} - \bar{\xi}^{(n)}_{t_m} \star \beta^{(n)}, \qquad \forall n, m. \tag{115}$$

To satisfy this condition we, first, assume that $\xi^{(m)}$ and $\bar{\xi}^{(m)}$ are expressed in terms of $\xi^{(1)}$ and $\bar{\xi}^{(1)}$ as follows:

$$\xi^{(m)}(\lambda,\mu;t) = \eta^{(m)}(\lambda,\nu) \star \xi^{(1)}(\nu,\mu), \quad m_1 + m_2 > 2, \qquad \eta^{(1)}(\lambda,\nu) = \mathcal{I}_2(\lambda,\nu), \tag{116}$$

$$\bar{\xi}^{(m)}(\lambda, \mu; t) = \bar{\xi}^{(1)}(\lambda, \nu; t) \star \bar{\eta}^{(m)}(\nu, \mu), \quad m_1 + m_2 > 2, \qquad \bar{\eta}^{(1)}(\nu, \mu) = \mathcal{I}_2(\lambda, \nu), \tag{117}$$

where $\eta^{(m)}(\nu, \mu)$ and $\bar{\eta}^{(m)}(\nu, \mu)$ are some $n_0 \times n_0$ matrix functions which will be specified below. Second, we define $\xi^{(1)}_{t_m}$ and $\bar{\xi}^{(1)}_{t_m}$ in terms of $\xi^{(1)}$ and $\bar{\xi}^{(1)}$ as follows:

$$\xi^{(1)}_{t_m}(\lambda, \mu; t) = T^{(m)}(\lambda, \nu) \star \xi^{(1)}(\nu, \mu; t), \tag{118}$$

$$\bar{\xi}^{(1)}_{t_m}(\lambda, \mu; t) = \bar{\xi}^{(1)}(\lambda, \nu; t) \star \bar{T}^{(m)}(\nu, \mu), \tag{119}$$

where $T^{(m)}(\lambda, \nu)$ and $\bar{T}^{(m)}(\lambda, \nu)$ are the $n_0 \times n_0$ matrix functions. At last, substitute equations (116)–(119) into equation (115). Since the resulting equation must be valid for any possible $\xi^{(1)}$ and $\bar{\xi}^{(1)}$, we obtain the following expressions for $\alpha^{(m)}$ and $\beta^{(m)}$:

$$\alpha^{(m)}(\lambda, \mu) = (\alpha^{(1)}(\lambda, \nu) * T^{(m)}(\nu, \tilde{\nu}) \star (T^{(1)})^{-1}(\tilde{\nu}, \nu)) \star (\eta^{(m)})^{-1}(\nu, \mu), \tag{120}$$

$$\beta^{(m)}(\lambda, \mu) = (\bar{\eta}^{(m)})^{-1}(\lambda, \nu) \star (\bar{T}^{(1)})^{-1}(\nu, \tilde{\nu}) \star \bar{T}^{(m)}(\tilde{\nu}, \tilde{\mu}) \star \beta^{(1)}(\tilde{\mu}, \mu). \tag{121}$$

In turn, compatibility of equations (118) and (119) requires

$$\begin{gathered}
T^{(m)} \star T^{(n)} - T^{(n)} \star T^{(m)} = 0 \quad \Rightarrow \quad T^{(m)}(\lambda, \mu) = (T(\lambda, \nu)\tau^{(m)}(\nu)) \star T^{-1}(\nu, \mu),\\
\bar{T}^{(m)} \star \bar{T}^{(n)} - \bar{T}^{(n)} \star \bar{T}^{(m)} = 0 \quad \Rightarrow \quad \bar{T}^{(m)}(\lambda, \mu) = \bar{T}^{-1}(\lambda, \nu) \star (\bar{\tau}^{(m)}(\nu)T(\nu, \mu)),\\
[\tau^{(m)}(\nu), \tau^{(n)}(\nu)] = 0, \qquad [\bar{\tau}^{(m)}(\nu), \bar{\tau}^{(n)}(\nu)] = 0,
\end{gathered} \tag{122}$$

where $T(\lambda, \nu)$, $\bar{T}(\lambda, \nu)$, $\tau^{(m)}(\nu)$ and $\bar{\tau}^{(m)}(\nu)$ are the $n_0 \times n_0$ matrix functions (compare with equation (40)). Thus, the compatibility condition of equation (112) generates equations (116)–(122).

Now we may integrate equations (118) and (119) obtaining the following expressions for $\xi^{(1)}$ and $\bar{\xi}^{(1)}$:

$$\xi^{(1)}(\lambda, \mu; t) = \left(T(\lambda, \nu)\, \mathrm{e}^{\sum_{i=1}^{D} \tau^{(i)}(\nu) t_i}\right) \star T^{-1}(\nu, \tilde{\nu}) \star \xi_0(\tilde{\nu}, \mu), \tag{123}$$

$$\bar{\xi}^{(1)}(\lambda, \mu; t) = \bar{\xi}_0(\lambda, \nu) \star \bar{T}^{-1}(\nu, \tilde{\nu}) \star \left(\mathrm{e}^{\sum_{i=1}^{D} \bar{\tau}^{(i)}(\tilde{\nu}) t_i}\, \bar{T}(\tilde{\nu}, \mu)\right). \tag{124}$$

Here $\xi_0(\tilde{\nu}, \mu)$ and $\bar{\xi}_0(\lambda, \nu)$ are the $n_0 \times n_0$ matrix functions. Finally, integrating equation (112) with $m_1 = m_2 = 1$, one obtains

$$\begin{aligned}
\chi(\lambda, \mu; t) = {} & \alpha^{(1)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star \xi^{(1)}(\tilde{\nu}, \mu)\\
& - \bar{\xi}^{(1)}(\lambda, \nu) \star (\bar{T}^{(1)})^{-1}(\nu, \tilde{\nu}) \star \beta^{(1)}(\tilde{\nu}, \mu).
\end{aligned} \tag{125}$$

It is convenient to rewrite equations (30) using equations (120) and (121) as follows:

$$A^{(m)}(\lambda, \mu) = \alpha^{(1)}(\lambda, \nu) \star \gamma^{(m)}(\nu, \tilde{\nu}) \star \beta^{(1)}(\tilde{\nu}, \mu), \tag{126}$$

$$\tilde{A}^{(m)}(\lambda) = \alpha^{(1)}(\lambda, \nu) \star \tilde{\Gamma}^{(m)}(\nu), \quad \forall m, \tag{127}$$

where

$$\begin{aligned}
\gamma^{(m)}(\lambda, \mu) = {} & T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu})(\eta^{(m)})^{-1}(\tilde{\nu}, \tilde{\mu})\\
& \star (\bar{\eta}^{(m)})^{-1}(\tilde{\mu}, \bar{\mu}) \star (\bar{T}^{(1)})^{-1}(\bar{\mu}, \tilde{\lambda}) \star \bar{T}^{(m)}(\tilde{\lambda}, \mu),
\end{aligned} \tag{128}$$

$$\tilde{\Gamma}^{(m)}(\lambda) = T^{(m)}(\lambda, \nu) \star (T^{(1)})^{-1}(\nu, \tilde{\nu}) \star (\eta^{(m)})^{-1}(\tilde{\nu}, \tilde{\mu}) * \tilde{\beta}^{(m)}(\tilde{\mu}). \tag{129}$$

Representations (126), (127) will be used in the rest of section 3.3.

3.3.1. Internal constraints for $\alpha^{(1)}$, $\beta^{(1)}$, $\tilde\Gamma^{(m)}$, T, $\bar T$, $\tau^{(m)}$ and $\bar\tau^{(m)}$. Note that the definitions of $\xi^{(m)}$ and $\bar\xi^{(m)}$ in terms of χ, i.e. equations (113) and (114), must be consistent with equation (125). This requirement generates some constraints for $\alpha^{(1)}$, $\beta^{(1)}$, $\tilde\Gamma^{(m)}$, T, $\bar T$, $\tau^{(m)}$ and $\bar\tau^{(m)}$. To derive constraints associated with definition (113), we apply the operator $(\beta^{(m)}+\tilde\beta^{(m)}P)*$ to equation (125) from the left obtaining the following equation:

$$\begin{aligned}\xi^{(m)}(\lambda,\mu;t) &= (\beta^{(m)}(\lambda,\nu)+\tilde\beta^{(m)}(\lambda)P(\nu))*\alpha^{(1)}(\nu,\tilde\nu)\star(T^{(1)})^{-1}(\tilde\nu,\tilde\mu)\star\xi^{(1)}(\tilde\mu,\mu)\\ &\quad-(\beta^{(m)}(\lambda,\nu)+\tilde\beta^{(m)}(\lambda)P(\nu))\star\bar\xi^{(1)}(\nu,\tilde\nu)*(T^{(1)})^{-1}(\tilde\nu,\tilde\mu)\star\beta^{(1)}(\tilde\mu,\mu).\end{aligned} \tag{130}$$

Substitute equation (116) for $\xi^{(m)}$ and require that the resulting equation is identity for any $\xi^{(1)}$. Then equation (130) becomes decomposed into the following two constraints:

$$(\beta^{(m)}(\lambda,\nu)+\tilde\beta^{(m)}(\lambda)P(\nu))*\bar\xi_0(\nu,\mu)=0, \tag{131}$$

$$\begin{aligned}\eta^{(m)}(\lambda,\mu) &= (\beta^{(m)}(\lambda,\nu)+\tilde\beta^{(m)}(\lambda)P(\nu))*\alpha^{(1)}(\nu,\tilde\nu)\star(T^{(1)})^{-1}(\tilde\nu,\mu),\\ &\quad m_i=1,\ldots,D_i,\qquad i=1,2.\end{aligned} \tag{132}$$

Similarly, to derive constraints associated with definition (114), we apply the operator $*\alpha^{(m)}$ to equation (125) from the right. One obtains

$$\bar\xi^{(m)}=\alpha^{(1)}\star(T^{(1)})^{-1}\star\xi^{(1)}*\alpha^{(m)}-\bar\xi^{(1)}\star(T^{(1)})^{-1}\star\beta^{(1)}*\alpha^{(m)}. \tag{133}$$

Substitute equation (117) for $\bar\xi^{(m)}$ and require that the resulting equation is identity for any $\bar\xi^{(1)}$. Then equation (133) becomes decomposed into following two constraints:

$$\xi_0(\lambda,\nu)*\alpha^{(1)}(\nu,\mu)=0, \tag{134}$$

$$\bar\eta^{(m)}(\lambda,\mu)=-(\bar T^{(1)})^{-1}(\lambda,\nu)\star\beta^{(1)}(\nu,\tilde\nu)*\alpha^{(m)}(\tilde\nu,\mu). \tag{135}$$

Equations (131) and (134) represent two constraints for $\alpha^{(1)}$, $\beta^{(m)}$ and $\tilde\beta^{(m)}$. Equations (132) and (135) with $m_1+m_2>2$ may be considered as definitions of $\eta^{(m)}$ and $\bar\eta^{(m)}$. However, since $\eta^{(1)}(\lambda,\mu)=\bar\eta^{(1)}(\lambda,\mu)=\mathcal{I}_2(\lambda,\mu)$, equation (135) with $m_1=m_2=1$ yields

$$\bar T^{(1)}(\lambda,\mu)=-\beta^{(1)}(\lambda,\nu)*\alpha^{(1)}(\nu,\mu). \tag{136}$$

In turn, equation (132) with $m_1=m_2=1$ in view of equation (136) yields

$$T^{(1)}(\lambda,\mu)=-\bar T^{(1)}(\lambda,\mu)+\tilde\beta^{(1)}(\lambda)P(\nu)*\alpha^{(1)}(\nu,\mu). \tag{137}$$

Both equations (136) and (137) may be treated as constraints for $T^{(1)}$, $\bar T^{(1)}$, $\alpha^{(1)}$ and $\beta^{(1)}$.

Now we simplify equation (135), $m_1+m_2>2$, replacing $\beta^{(1)}\star\alpha^{(1)}$ in accordance with equation (136). One obtains

$$\bar\eta^{(m)}\star\eta^{(m)}=T^{(m)}\star(T^{(1)})^{-1}\quad\overset{\text{equation (128)}}{\Rightarrow} \tag{138}$$

$$\gamma^{(m)}(\lambda,\mu)=(\bar T^{(1)})^{-1}(\lambda,\nu)\star\bar T^{(m)}(\nu,\mu),\qquad m_1+m_2>2, \tag{139}$$

which is the definition of $\gamma^{(m)}$. Deriving equation (139) we assume invertibility of the operator $\star T^{(m)}$, $\forall m$.

Constraints (131) and (132) may be simplified multiplying them by $\bar\eta^{(m)}$ from the left, using definitions of $\alpha^{(m)}$ (120), $\beta^{(m)}$ (121), $\tilde\Gamma^{(m)}$ (129) and eliminating $\bar\eta^{(m)}\star\eta^{(m)}$ with equation (138). We obtain in result

$$((\bar T^{(1)})^{-1}(\lambda,\nu)\star\bar T^{(m)}(\nu,\tilde\nu)\star\beta^{(1)}(\tilde\nu,\tilde\mu)+\tilde\Gamma^{(m)}(\lambda)P(\tilde\mu))*\bar\xi_0(\tilde\mu,\mu)=0,\quad\forall m, \tag{140}$$

$$\bar\eta^{(m)} \star \tilde\beta^{(m)} P * \alpha^{(1)} = T^{(m)} + \bar T^{(m)} \quad \overset{\text{equation (129)}}{\Rightarrow} \tag{141}$$

$$\tilde\Gamma^{(m)}(\lambda) P(\nu) * \alpha^{(1)}(\nu,\mu) = T^{(m)}(\lambda,\mu) + \bar T^{(m)}(\lambda,\mu), \qquad m_1 + m_2 > 2. \tag{142}$$

One has to take into account that $T^{(m)}$, $\bar T^{(m)}$ are represented by equations (122) in terms of $T(\lambda,\mu)$, $\bar T(\lambda,\mu)$, $\tau^{(m)}(\lambda,\mu)$ and $\bar\tau^{(m)}(\lambda,\mu)$. Thus, we have obtained a set of constraints for the functions $\alpha^{(1)}$, $\beta^{(1)}$, $\tilde\Gamma^{(m)}$, T, $\bar T$, $\tau^{(m)}$ and $\bar\tau^{(m)}$: equations (134), (136), (137), (140), (142).

Note that, similar to section 2.3.1, all these constraints are generated by the system (95) and its compatibility condition. For this reason we refer to them as the internal constraints in order to defer them from so-called external constraints which will be introduced 'by hand' for the purpose of derivation of the nonlinear PDEs, see theorems 3.3, 3,4, 3,5.

3.3.2. System of compatible linear equations for $W(\lambda;t)$.

Theorem 3.3. *Let the matrices $A^{(m)}(\lambda,\mu)$ satisfy the following external constraint:*

$$\sum_{m_1=1}^{D_1} L^{(m_1)} P(\lambda) * A^{(m)}(\lambda,\mu) = S^{(m_2)} P(\lambda), \tag{143}$$

where $L^{(m_1)}$ and $S^{(m_2)}$ are some $n_0 \times n_0$ constant matrices. Then the matrix function $W(\lambda;t)$ obtained as a solution to the integral equation (11) with χ defined by equation (95) is a solution to the following system of compatible linear equations:

$$\begin{aligned} E^{(m_2)}(\lambda;t) &:= \sum_{m_1=1}^{D_1} L^{(m_1)} \left(W_{t_m}(\lambda;t) + V^{(m)}(t) W(\lambda;t) + W(\mu;t) * A^{(m)}(\mu,\lambda;t)\right) \\ &= S^{(m_2)} W(\lambda;t), \qquad m_i = 1,\ldots,D_i, \quad i = 1,2, \end{aligned} \tag{144}$$

where $V^{(m)}$ is given by equation (16).

Proof. To derive equation (144), we differentiate equation (11) with respect to t_m. Then, in view of equation (95), one obtains the following integral equation:

$$\begin{aligned} &\mathcal{E}^{(m)}(\mu;t) := P(\nu) * A^{(m)}(\nu,\lambda) * \chi(\lambda,\mu;t) = \tilde E^{(m)}(\nu;t) * (\chi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)), \\ &\tilde E^{(m)}(\mu;t) = W_{t_m}(\mu;t) + V^{(m)}(t) W(\mu;t) + W(\nu;t) * A^{(m)}(\nu,\mu). \end{aligned} \tag{145}$$

Consider the following combination of equations (145): $\sum_{m_1=1}^{D_1} L^{(m_1)} \mathcal{E}^m$. Then, using constraint (143), one obtains in result

$$\begin{aligned} \sum_{m_1=1}^{D_1} L^{(m_1)} \mathcal{E}^m &:= \sum_{m_1=1}^{D_1} L^{(m_1)} \tilde E^{(m)}(\nu;t) * (\chi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)) \\ &= S^{(m_2)} W(\nu;t) * (\chi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)). \end{aligned} \tag{146}$$

Since the operator $*(\chi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu))$ is invertable, equation (146) is equivalent to equation (144). □

Remark. Similar to equation (15), equation (144) is, strictly speaking, a nonlinear equation for $W(\lambda;t)$ since $V^{(m)}(t)$ are defined by equation (16) in terms of $W(\lambda;t)$.

The system (144) is an analogy of the overdetermined system of linear equations in the classical inverse spectral transform method. According to this method, nonlinear PDEs for potentials of the overdetermined linear system appear as compatibility conditions for this system. However, nonlinear PDEs may not be obtained by this method in our case because of the last term on the lhs of equation (144). Instead of this, we represent another algorithm of derivation of the nonlinear PDEs in sections 3.3.3 and 3.3.4.

3.3.3. First-order nonlinear PDEs for the fields $V^{(m)}(t)$, $m_i = 1, \ldots, D_i$, $i = 1, 2$.

Theorem 3.4. *In addition to equations (11), (95) and external constraint (143), we impose one more external constraint:*

$$\sum_{m_2=1}^{D_2} A^{(m)}(\lambda, \nu) * \tilde{A}^{(n)}(\nu) R^{(m_2 n)} = \sum_{j=1}^{D} \tilde{A}^{(j)}(\lambda) P^{(m_1 j n)}, \qquad n_i = 1, \ldots, D_i, \quad i = 1, 2, \tag{147}$$

where $R^{(m_2 n)}$ and $P^{(m_1 j n)}$ are some $n_0 \times n_0$ constant matrices. Then the $n_0 \times n_0$ matrix functions $V^{(m)}(t)$ are solutions to the following system of nonlinear PDEs:

$$\sum_{m=1}^{D} L^{(m_1)}\big(V^{(n)}_{t_m} + V^{(m)} V^{(n)} + V^{(m)} \mathcal{A}^{(n)}\big) R^{(m_2 n)} + \sum_{m_1=1}^{D_1} L^{(m_1)} \sum_{j=1}^{D} V^{(j)} P^{(m_1 j n)}$$
$$= \sum_{m_2=1}^{D_2} S^{(m_2)} V^{(n)} R^{(m_2 n)}, \tag{148}$$
$$\mathcal{A}^{(n)} = P(\lambda, \nu) * \tilde{A}^{(n)}(\nu), \qquad n_i = 1, \ldots, D_i, \quad i = 1, 2.$$

Proof. Applying the operator $*\tilde{A}^{(n)}$ to equation (144) from the right one obtains the following equation:

$$E^{(m_2 n)}(t) = E^{(m_2)}(\lambda; t) * \tilde{A}^{(n)}(\lambda)$$
$$:= \sum_{m_1=1}^{D_1} L^{(m_1)}\big(V^{(n)}_{t_m} + V^{(m)} V^{(n)} + V^{(m)} \mathcal{A}^{(n)} + U^{(mn)}\big) = \sum_{m_2=1}^{D_2} S^{(m_2)} V^{(n)}, \tag{149}$$

which introduces a new set of fields $U^{(mn)}$, see equation (52). Due to the constraint (147), we may eliminate these fields using a proper combination of equations (149). Namely, combination $\sum_{m_2=1}^{D_2} E^{(m_2 n)} R^{(m_2 n)}$ results in the system (148). □

Reduction 1. The derived equation (148) admits the following reduction for its coefficients:

$$P^{(m_1 j n)} = -\mathcal{A}^{(n)} R^{(j_2 n)} \delta_{j_1 m_1}, \qquad S^{(m_2)} = 0. \tag{150}$$

Then equation (148) reads

$$\sum_{m=1}^{D} L^{(m_1)}\big(V^{(n)}_{t_m} + V^{(m)} V^{(n)}\big) R^{(m_2 n)} = 0, \tag{151}$$

which is represented in the introduction, see equation (5). This reduction does not effect constraint (143) while constraint (147) reads

$$\sum_{m_2=1}^{D_2} (A^{(m)}(\lambda, \nu) + \tilde{A}^{(m)}(\lambda) P(\nu)) * \tilde{A}^{(n)}(\nu) R^{(m_2 n)} = 0, \qquad n_i = 1, \ldots, D_i, \quad i = 1, 2. \tag{152}$$

Reduction 2. Along with reduction (150), we consider reduction (56), (57) with

$$C^{(n)} R^{(m_2 n)} = R^{(m_2)}, \tag{153}$$

where $C^{(m)}$ and $R^{(m_2)}$ are some $n_0 \times n_0$ constant matrices. Then the system (151) reduces to the single PDE:

$$\sum_{m=1}^{D} L^{(m_1)}\big(V_{t_m} + VC^{(m)}V\big)R^{(m_2)} = 0, \tag{154}$$

which is written in the introduction, see equation (5). Reduction (153) does not effect constraint (143), while constraint (152) reduces to the following single equation:

$$\sum_{m_2=1}^{D_2} (A^{(m)}(\lambda,\nu) + \tilde{A}(\lambda)C^{(m)}P(\nu)) * \tilde{A}(\nu)R^{(m_2)} = 0. \tag{155}$$

If, in addition, $C^{(m_1m_2)}$, $L^{(m_1)}$ and $R^{(m_2)}$ are diagonal,

$$\begin{aligned} &C^{(m_1m_2)} = -C^{(m_2m_1)}, && R^{(m_2)} \equiv L^{(m_2)},\\ &V_{t_{m_1m_2}} = -V_{t_{m_2m_1}}, && D_1 = D_2 = D_0, \end{aligned} \tag{156}$$

then nonlinear equation (154) reduces to the following equation:

$$\begin{aligned} \sum_{\substack{m_1,m_2=1\\ m_2>m_1}}^{D_0} &\big(L^{(m_1)}V_{t_{m_1m_2}}L^{(m_2)} - L^{(m_2)}V_{t_{m_1m_2}}L^{(m_1)}\\ &+ L^{(m_1)}VC^{(m_1m_2)}VL^{(m_2)} - L^{(m_2)}VC^{(m_1m_2)}VL^{(m_1)}\big) = 0, \end{aligned} \tag{157}$$

which is a multidimensional generalization of the classical (2+1)-dimensional S-integrable N-wave equation (105).

Equation (157) admits reduction

$$t_{m_1m_2} = -\mathrm{i}\tau_{m_1m_2}, \qquad V = -V^+, \quad m_2 > m_1, \tag{158}$$

see equation (8), which is important for physical applications.

3.3.4. Second-order nonlinear PDEs for $V^{(m)}(t)$, $m_i = 1, \ldots, D_i$, $i = 1, 2$.

Theorem 3.5. *In addition to equations (11), (95) and external constraint (143) we impose one more external constraint on the matrix functions $A^{(m)}(\lambda,\nu)$ and $\tilde{A}^{(m)}(\lambda)$ (instead of constraint (147)):*

$$\begin{aligned} &\sum_{n=1}^{D}\sum_{m_2=1}^{D_2} A^{(m)}(\lambda,\nu) * A^{(n)}(\nu,\mu) * \tilde{A}^{(l)}(\mu)R^{(m_2nl)}\\ &\quad = \sum_{n,p=1}^{D} A^{(n)}(\lambda,\mu) * \tilde{A}^{(p)}(\mu)P^{(m_1npl)} + \sum_{n=1}^{D} \tilde{A}^{(n)}(\lambda)P^{(m_1nl)}, \end{aligned} \tag{159}$$

where $R^{(m_2nl)}$, $P^{(m_1npl)}$ and $P^{(m_1nl)}$ are some $n_0 \times n_0$ constant matrices. Let $S^{(m_2)} = 0$ for the sake of simplicity. Then the $n_0 \times n_0$ matrix functions $V^{(m)}(t)$ are solutions to the following system of nonlinear PDEs:

$$\begin{aligned} &\sum_{m,n=1}^{D} L^{(m_1)}\big(U^{(nl)}_{t_m} + V^{(m)}U^{(nl)} + V^{(m)}\mathcal{A}^{(nl)}\big)R^{(m_2nl)}\\ &\quad + \sum_{m_1=1}^{D_1} L^{(m_1)}\left(\sum_{n,p=1}^{D} U^{(np)}P^{(m_1npl)} + \sum_{n=1}^{D} V^{(n)}P^{(m_1nl)}\right) = 0, \end{aligned} \tag{160}$$

$$\mathcal{A}^{(nl)} = P(\lambda) * A^{(n)}(\lambda,\mu) * \tilde{A}^{(l)}(\mu), \qquad l_i = 1, \ldots, D_i, \quad i = 1, 2,$$

where fields $U^{(mn)}$ are related with $V^{(m)}$ due to equation (149):

$$\begin{aligned}
&\sum_{m_1=1}^{D_1} L^{(m_1)}\big(V^{(n)}_{t_m} + V^{(m)}V^{(n)} + V^{(m)}\mathcal{A}^{(n)} + U^{(mn)}\big) = 0,\\
&\mathcal{A}^{(n)} = P(\lambda) * \tilde{A}^{(n)}(\lambda).
\end{aligned} \tag{161}$$

Proof. First of all, applying the operator $*\tilde{A}^{(n)}$ to equation (144) from the right one obtains equation (161), which introduces a new set of fields $U^{(mn)}(t)$, see equation (52). This step is equivalent to the first step in derivation of equation (148). However, we may not eliminate these fields from the system (161) because constraint (147) is not valid in this theorem. Instead of this, we derive nonlinear equations for fields $U^{(mn)}(t)$ applying the operator $*A^{(n)} * \tilde{A}^{(l)}$ to equation (144) from the right. One obtains

$$\begin{aligned}
E^{(m_2nl)}(t) &= E^{(m_2)}(\lambda;t) * A^{(n)}(\lambda,\mu) * \tilde{A}^{(l)}(\mu)\\
&:= \sum_{m_1=1}^{D_1} L^{(m_1)}\big(U^{(nl)}_{t_m} + V^{(m)}U^{(nl)} + V^{(m)}\mathcal{A}^{(nl)} + U^{(mnl)}\big) = 0,
\end{aligned} \tag{162}$$

where the fields $U^{(mnl)}$ are defined by equation (65). Due to constraint (159), these fields may be eliminated taking a proper combination of equations (162), namely $\sum_{m_2=1}^{D_2} E^{(m_2nl)}R^{(m_2nl)}$. In result, one obtains equation (160). □

Reduction 1. Let

$$\begin{aligned}
&P^{(m_1npl)} = \delta_{m_1n_1}\tilde{P}^{(n_2pl)}, \qquad P^{(m_1nl)} = \delta_{m_1n_1}\tilde{P}^{(n_2l)},\\
&\tilde{P}^{(n_2l)} = \sum_{p=1}^{D}(\mathcal{A}^{(p)}\tilde{P}^{(n_2pl)} - \mathcal{A}^{(pl)}R^{(n_2pl)}).
\end{aligned} \tag{163}$$

Then equation (160) reduces to the following one:

$$\begin{aligned}
&\sum_{m,n=1}^{D} L^{(m_1)}\big[\big(U^{(nl)}_{t_m} + V^{(m)}U^{(nl)}\big)R^{(m_2nl)} + \big(V^{(n)}_{t_m} + V^{(m)}V^{(n)}\big)\tilde{P}^{(m_2nl)}\big] = 0,\\
&\qquad l_i = 1,\dots,D_i, \qquad i = 1,2.
\end{aligned} \tag{164}$$

Constraint (143) remains the same (with $S^{(m_2)} = 0$) while constraint (159) reads

$$\begin{aligned}
&\sum_{n=1}^{D}\sum_{m_2=1}^{D_2}(A^{(m)}(\lambda,\nu) + \tilde{A}^{(m)}(\lambda)P(\nu)) * (A^{(n)}(\nu,\mu) * \tilde{A}^{(l)}(\mu)R^{(m_2nl)} - \tilde{A}^{(n)}(\tilde{\nu})\tilde{P}^{(m_2nl)}),\\
&\qquad m_1 = 1,\dots,D_1, \qquad l_i = 1,\dots,D_i, \quad i = 1,2.
\end{aligned} \tag{165}$$

Reduction 2. Along with reduction (163) we consider reduction (56), (57) together with the following conditions:

$$U^{(ml)}(t) = U^{(m)}(t)C^{(l)}, \qquad C^{(l)}R^{(m_2nl)} = R^{(m_2n)}, \qquad C^{(n)}\tilde{P}^{(m_2nl)} = P^{(m_2n)}. \tag{166}$$

Then the system (164) reduces to the following single PDE:

$$\sum_{m,n=1}^{D} L^{(m_1)}\big[\big(U^{(n)}_{t_m} + VC^{(m)}U^{(n)}\big)R^{(m_2n)} + \big(V_{t_m} + VC^{(m)}V\big)P^{(m_2n)}\big] = 0. \tag{167}$$

This reduction does not effect constraint (143), while constraint (165) reduces to the following single equation:

$$\sum_{m_2=1}^{D_2}\sum_{n=1}^{D}(A^{(m)}(\lambda,\nu)+\tilde{A}(\lambda)C^{(m)}P(\nu))*(A^{(n)}(\nu,\mu)*\tilde{A}(\mu)R^{(m_2n)}-\tilde{A}(\nu)P^{(m_2n)}). \qquad (168)$$

Emphasize that systems (148) and (160), (161) do not represent commuting flows since constraints (147) and (159) are not compatible in general.

3.3.5. Solutions to the first-order equation (154).

On the dimensionality of the available solution space. The important question is whether the derived nonlinear DPEs are completely integrable. In other words, regarding equation (154), is it possible to introduce a proper number of arbitrary functions of D_1D_2-1 variables in the solution space.

In order to clarify this problem, first of all, let us consider small χ. Then, using equation (11), we approximate V by equation (72). If $\tau^{(m)}(\nu)$ and $\bar{\tau}^{(m)}(\nu)$ are the arbitrary functions of arguments, then the above expression for V involves two arbitrary functions of D_1D_2 variables t_m, $m_i=1,\ldots,D_i$, $i=1,2$:

$$\begin{aligned}F_1(t)&=P*\alpha^{(1)}\star(T^{(1)})^{-1}\star\xi^{(1)}*\alpha^{(1)}\star\tilde{\Gamma}=\int g_{11}(\nu)\,\mathrm{e}^{\sum_{i=1}^{D}\tau^{(i)}t_i}g_{12}(\nu)\,\mathrm{d}\Omega_2(\nu),\\ F_2(t)&=P*\alpha^{(1)}*\bar{\xi}^{(1)}\star(\bar{T}^{(1)})^{-1}\star\beta^{(1)}*\alpha^{(1)}\star\tilde{\Gamma}=\int g_{21}(\nu)\,\mathrm{e}^{\sum_{i=1}^{D}\tau^{(i)}t_i}g_{22}(\nu)\,\mathrm{d}\Omega_2(\nu),\end{aligned} \qquad (169)$$

where

$$\begin{aligned}g_{11}&=P*\alpha^{(1)}\star(T^{(1)})^{-1}\star T, \qquad & g_{12}&=T^{-1}\star\xi_0*\alpha^{(1)}\star\tilde{\Gamma},\\ g_{21}&=P*\alpha^{(1)}\star\bar{\xi}_0\star\bar{T}^{-1}, \qquad & g_{22}&=\bar{T}\star(\bar{T}^{(1)})^{-1}\star\beta^{(1)}*\alpha^{(1)}\star\tilde{\Gamma}.\end{aligned} \qquad (170)$$

However, not all $\tau^{(m)}(\nu)$ and $\bar{\tau}^{(m)}(\nu)$ are arbitrary since we have to resolve constraints (134), (136), (137), (140), (142), (143), (155). Constraints (134), (140) may be satisfied using a special structure of $\alpha^{(1)},\beta^{(1)}$, P, $\bar{\xi}_0,\xi_0$ (see the example of an explicit solution, equation (182)), which does not reduce the dimensionality of the solution space. Constraints (136) may be considered as constraints for $\alpha^{(1)}$ and $\beta^{(1)}$. Constraints (137), (142) relate $\bar{T}^{(m)}$ with $T^{(m)}$ and $\tilde{\Gamma}$, so that, generally speaking, only one of functions (169) remains arbitrary. The remaining two constraints (143), (155) introduce D_2 and D_1 relations among the parameters $\bar{\tau}^{(m)}$, $m_i=1,\ldots,D_i$, $i=1,2$, which, in general, reduces significantly the dimensionality of the solution space. However, this depends on the particular choice of the coefficients $L^{(mj_1)}$, $R^{(m_2)}$ and $C^{(m)}$.

This is a preliminary analysis which suggests us to look for examples of completely integrable PDEs in the derived class of new nonlinear PDEs.

Construction of explicit solutions. We construct explicit solutions in the form of rational functions of exponents for the first-order nonlinear PDE (154) with $\mathrm{d}\Omega_i(\nu)$, $i=1,2$, given by equation (74). Along with notations (75) we use the following ones:

$$\begin{aligned}&\hat{\bar{\xi}}_0=\begin{bmatrix}\bar{\xi}_0(a_1,b_1)&\cdots&\bar{\xi}_0(a_1,b_N)\\ \cdots&\cdots&\cdots\\ \bar{\xi}_0(a_M,b_1)&\cdots&\bar{\xi}_0(a_M,b_N)\end{bmatrix}, \qquad \hat{\bar{\tau}}^{(m)}=\mathrm{diag}(\tau^{(m)}(b_1),\cdots,\tau^{(m)}(b_N)),\\ &T(\lambda,\mu)=\bar{T}(\lambda,\mu)=\mathcal{I}_2(\lambda,\mu).\end{aligned} \qquad (171)$$

Field $V(t)$ is represented by equation (78) where the function χ is given by equation (125). In turn, the functions $\xi^{(1)}$ and $\bar{\xi}^{(1)}$ are given by equations (123) and (124). In result, using notations (75), (171), we obtain the following formula for $\hat{\chi}$:

$$\hat{\chi} = \hat{\alpha}^{(1)}(\hat{\tau}^{(1)})^{-1}\,\mathrm{e}^{\sum_{i=1}^{D}\hat{\tau}^{(i)}t_i}\hat{\xi}^{(0)} + \hat{\hat{\xi}}^{(0)}\,\mathrm{e}^{\sum_{i=1}^{D}\hat{\hat{\tau}}^{(i)}t_i}(\hat{\hat{\tau}}^{(1)})^{-1}\hat{\beta}^{(1)}. \tag{172}$$

Finally, constraints (134), (136), (137), (140), (142), (143), (155) must be satisfied, which read in view of notations (75), (171) as follows, $m_i = 1, \ldots, D_i$, $i = 1, 2$:

$$\hat{\xi}_0\hat{\alpha}^{(1)} = 0, \tag{173}$$

$$\hat{\hat{\tau}}^{(1)} = -\hat{\beta}^{(1)}\hat{\alpha}^{(1)}, \tag{174}$$

$$\hat{\hat{\Gamma}}C^{(m)}\hat{P}\hat{\alpha}^{(1)} = \hat{\tau}^{(m)} + \hat{\hat{\tau}}^{(m)}, \tag{175}$$

$$((\hat{\hat{\tau}}^{(1)})^{-1}\hat{\hat{\tau}}^{(m)}\hat{\beta}^{(1)} + \hat{\hat{\Gamma}}C^{(m)}\hat{P})\hat{\hat{\xi}}_0 = 0, \tag{176}$$

$$\sum_{m_1=1}^{D_1} L^{(m_1)}\hat{P}\hat{\alpha}^{(1)}(\hat{\hat{\tau}}^{(1)})^{-1}\hat{\hat{\tau}}^{(m)}\hat{\beta}^{(1)} = 0, \tag{177}$$

$$\sum_{m_2=1}^{D_2} \hat{\alpha}^{(1)}((\hat{\hat{\tau}}^{(1)})^{-1}\hat{\hat{\tau}}^{(m)}\beta^{(1)} + \hat{\hat{\Gamma}}C^{(m)}\hat{P})\hat{\alpha}^{(1)}\hat{\hat{\Gamma}}R^{(m_2)} = 0. \tag{178}$$

Here we combine equations (137) and (142) into the single equation (175). Let us transform some of equations (173)–(178). For instance, considering only particular solutions to equation (176) we assume

$$\hat{\beta}^{(1)}\hat{\hat{\xi}}_0 = \hat{P}\hat{\hat{\xi}}_0 = 0. \tag{179}$$

In other words, the rows of $\hat{\beta}^{(1)}$ and $\hat{P}$ are orthogonal to the columns of $\hat{\hat{\xi}}_0$. Similarly, equation (173) means that the rows of $\hat{\xi}_0$ are orthogonal to the columns of $\hat{\alpha}^{(1)}$.

In turn, constraint (178) may be transformed into the following one using equation (174):

$$\sum_{m_2=1}^{D_2} \hat{\alpha}^{(1)}(\hat{\hat{\tau}}^{(m)} - \hat{\hat{\Gamma}}C^{(m)}\hat{P}\hat{\alpha}^{(1)})\hat{\hat{\Gamma}}R^{(m_2)} = 0. \tag{180}$$

Since $\hat{\hat{\tau}}^{(1)}$ is invertable, one requires $M > N$, which is evident due to equation (174). Then we may rewrite equation (180) without factor $\hat{\alpha}^{(1)}$ as follows:

$$\sum_{m_2=1}^{D_2} (\hat{\hat{\tau}}^{(m)} - \hat{\hat{\Gamma}}C^{(m)}\hat{P}\alpha^{(1)})\hat{\hat{\Gamma}}R^{(m_2)} = 0. \tag{181}$$

All in all, the following constraints must be satisfied: (173)–(175), (177), (179), (181).

Simple examples of explicit solutions. We obtain a particular solution to the four-dimensional nonlinear PDE (154), i.e. $D_1 = D_2 = 2$. Remember that we use double indices so that, for instance, $\hat{\hat{\beta}}^{(m)} = \hat{\hat{\beta}}^{(m_1m_2)}$. Let $M = 5, N = 2, n_0 = 2, L^{(1)} = R^{(1)} = I, L^{(2)} = \mathrm{diag}(l_1, l_2)$, $R^{(2)} = \mathrm{diag}(r_1, r_2)$, $C^{(n_1n_2)} = \mathrm{diag}\big(c_1^{(n_1n_2)}, c_2^{(n_1n_2)}\big)$, $c_1^{(11)} = c_2^{(11)} = 1$. To satisfy equations (173) and (179) we use the following structures of matrices:

$$\hat{\beta}^{(11)} = \begin{bmatrix} Z & Z & I & Z & J_1 \\ Z & Z & Z & J_1 & I \end{bmatrix}, \qquad \hat{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & -1 & 2 & 1 & 2 \end{bmatrix},$$

$$\hat{\hat{\Gamma}}^{(11)} = \begin{bmatrix} J_0 \\ Z \end{bmatrix}, \qquad \hat{\xi}_0 = \begin{bmatrix} J_0 & Z & Z & Z & Z \\ Z & -J_2 & Z & Z & Z \end{bmatrix}, \tag{182}$$

$$\hat{\hat{\xi}}_0 = \begin{bmatrix} K_2 & Z \\ Z & J_0 \\ Z & Z \\ Z & Z \\ Z & Z \end{bmatrix}, \qquad \hat{\alpha}^{(11)} = \begin{bmatrix} Z_4 \\ K_{6,4} \end{bmatrix},$$

where Z and I are the 2×2 zero and identity matrices, respectively, Z_4 is a 4×4 zero matrix, $K_{6,4}$ is a 6×4 constant matrix which will be defined below:

$$J_0 = \mathrm{diag}(1, -1), \qquad J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} s_3 & 1 \\ s_4 & s_5 \end{bmatrix}. \tag{183}$$

We take $\hat{\hat{\tau}}^{(11)} = \mathrm{diag}(1, 2, 3, 4)$.

To satisfy equations (174) and (175) ($m_1 = m_2 = 1$) we take

$$K_{6,4} = \begin{bmatrix} \frac{1}{5}(2s_1 - 3) & s_2 & -1 & \frac{4}{3} \\ s_1 & s_2 - 2 & \frac{1}{2} & \frac{4}{3} \\ \frac{2}{5}(s_1 + 1) & s_2 & -1 & -\frac{8}{3} \\ s_1 & s_2 & -\frac{5}{2} & \frac{4}{3} \\ -s_1 & -s_2 & -\frac{1}{2} & -\frac{4}{3} \\ -\frac{2}{5}(s_1 + 1) & -s_2 & 1 & -\frac{4}{3} \end{bmatrix},$$

$$\tau^{(11)} = \mathrm{diag}\left(\frac{2}{5}(3s_1 - 2,\ 3s_2,\ -3,\ -4\right). \tag{184}$$

To satisfy equations (177) and (181), $m_1+m_2 > 2$, we take the matrices $\hat{\Gamma}^{(n_1 n_2)}$ in the following form:

$$\hat{\hat{\Gamma}}^{(12)} = \begin{bmatrix} J_3 \\ Z \end{bmatrix}, \qquad \hat{\hat{\Gamma}}^{(21)} = \begin{bmatrix} J_4 \\ Z \end{bmatrix}, \qquad \hat{\hat{\Gamma}}^{(22)} = \begin{bmatrix} J_5 \\ Z \end{bmatrix},$$

$$J_3 = \mathrm{diag}\left(-\frac{1}{r_1}\big(1 + c_1^{(21)} l_1 + c_1^{(22)} l_1 r_1\big), \frac{1}{r_2}\big(1 + c_2^{(21)} l_2 + c_2^{(22)} l_2 r_2\big)\right), \tag{185}$$

$$J_4 = \mathrm{diag}\big(c_1^{(21)}, -c_2^{(21)}\big), \qquad J_5 = \mathrm{diag}\big(c_1^{(22)}, -c_2^{(22)}\big).$$

Elements of the matrix $C^{(12)}$ must be defined as follows:

$$c_i^{(12)} = -\frac{1}{r_i}\big(1 + l_i\big(c_i^{(21)} + c_i^{(22)} r_i\big)\big), \qquad i = 1, 2, \tag{186}$$

and $\hat\tau^{(n_1,n_2)}$, $\hat{\hat\tau}^{(n_1,n_2)}$ are as follows:

$$\begin{aligned}
\hat\tau^{(12)} &= \mathrm{diag}\left(\frac{4-6s_1}{5r_1},\ -\frac{3s_2}{r_2},\ \hat\tau_3^{(12)},\ \hat\tau_4^{(12)}\right),\\
\hat\tau^{(21)} &= \mathrm{diag}\left(\frac{1}{5}c_1^{(21)}(6s_1+1)+\frac{1}{l_1},\ c_2^{(21)}(2+3s_2)+\frac{2}{l_2},\ \hat\tau_3^{(21)},\ \hat\tau_4^{(21)}\right),\\
\hat\tau^{(22)} &= \mathrm{diag}\left(-\frac{5+c_1^{(21)}(1+6s_1)l_1}{5l_1r_1},\ -\frac{2+c_2^{(21)}(2+3s_2)l_2}{l_2r_2},\ \hat\tau_3^{(22)},\ \hat\tau_4^{(22)}\right),\\
\bar\tau^{(12)} &= \mathrm{diag}\left(-\frac{1}{5r_1}\big(5+(1+6s_1)l_1\big(c_1^{(21)}+c_1^{(22)}r_1\big)\big),\right.\\
&\qquad\left. -\frac{1}{r_2}\big(2+(2+3s_2)l_2\big(c_2^{(21)}+c_2^{(22)}r_2\big)\big), -\tau_3^{(12)}, -\tau_4^{(12)}\right),\\
\hat{\hat\tau}^{(21)} &= -\mathrm{diag}\left(\frac{1}{l_1},\ \frac{2}{l_2},\ \tau_3^{(21)},\ \tau_4^{(21)}\right),\\
\hat{\hat\tau}^{(22)} &= \mathrm{diag}\left(\frac{1}{5l_1r_1}\big(5+(1+6s_1)l_1\big(c_1^{(21)}+c_1^{(22)}r_1\big)\big),\right.\\
&\left.\frac{1}{l_2r_2}\big(2+(2+3s_2)l_2\big(c_2^{(21)}+c_2^{(22)}r_2\big)\big), -\hat\tau_3^{(22)}, -\hat\tau_4^{(22)}\right).
\end{aligned}\tag{187}$$

Introduce the positive parameters p_i, $i = 1, 2, 3, 4$, by the following formulas:

$$\begin{aligned}
s_2 &= \frac{p_3}{18s_1-12}, & s_3 &= -\frac{2}{5p_3}(3s_1-2)p_1,\\
s_4 &= \frac{1}{5p_3}(p_3p_4-p_1p_2), & s_5 &= \frac{p_2}{6s_1-4},
\end{aligned}\tag{188}$$

so that the solution $V(t)$ reads

$$\begin{aligned}
V(t) &= \frac{1}{D}\begin{bmatrix} f_{11}(p_2\,\mathrm{e}^{\hat\eta_2}+p_3\,\mathrm{e}^{\hat\eta_1+\hat\eta_2}) & f_{12}\,\mathrm{e}^{\hat\eta_3}\\ f_{21}\,\mathrm{e}^{\hat\eta_1+\hat\eta_2-\hat\eta_3} & f_{22}(p_1\mathrm{e}^{\hat\eta_1}+p_3\,\mathrm{e}^{\hat\eta_1+\hat\eta_2})\end{bmatrix},\\
D &= p_1\mathrm{e}^{\hat\eta_1}+p_2\,\mathrm{e}^{\hat\eta_2}+p_3\,\mathrm{e}^{\hat\eta_1+\hat\eta_2}+p_4,\\
f_{11} &= -\frac{1}{5}(1+6s_1),\qquad f_{12} = \frac{p_3(1+6s_1)}{2(3s_1-2)},\\
f_{21} &= \frac{1}{5p_3}(p_1p_2-p_3p_4)(12s_1+p_3-8),\qquad f_{22} = -\frac{12s_1+p_3-8}{2(3s_1-2)}.
\end{aligned}\tag{189}$$

Here

$$\begin{aligned}
&\hat\eta_1 = \eta_1-\eta_3,\qquad \hat\eta_2 = \eta_2-\eta_3,\qquad \hat\eta_3 = \eta_4-\eta_3,\\
&\eta_i = \sum_{m_1,m_2=1}^{2} a_{m_1m_2}^{(i)}t_{m_1m_2},\qquad i = 1, 2, 3, 4,
\end{aligned}\tag{190}$$

$$
\begin{aligned}
a_{11}^{(1)} &= \frac{1+6s_1}{5}, \quad a_{12}^{(1)} = \frac{1}{2r_2}\left(4+\frac{p_3}{3s_1-2}\right),\\
a_{21}^{(1)} &= \frac{1}{5l_2}\big(10+l_2c_1^{(21)}(1+6s_1)\big),\\
a_{22}^{(1)} &= \frac{1}{5}\left(\frac{5+l_1c_1^{(21)}}{l_1r_1}-\frac{10c_2^{(21)}}{r_2}+c_1^{(22)}-10c_2^{(22)}\right.\\
&\quad\left.+\frac{5\big(c_2^{(21)}+r_2c_2^{(22)}\big)p_3}{r_2(4-6s_1)}+\frac{6\big(c_1^{(21)}+r_1c_1^{(22)}\big)s_1}{r_1}\right),
\end{aligned}
\tag{191}
$$

$$
\begin{aligned}
a_{11}^{(2)} &= 2+\frac{p_3}{6s_1-4},\\
a_{12}^{(2)} &= \frac{1}{5}\left(\frac{\big(1+l_1c_1^{(21)}\big)(1+6s_1)}{r_1}+l_1c_1^{(22)}(1+6s_1)\right.\\
&\quad\left.-\frac{5l_2\big(c_2^{(21)}+r_2c_2^{(22)}\big)(-8+p_3+12s_1)}{2r_2(3s_1-2)}\right),\\
a_{21}^{(2)} &= \frac{2}{l_2}+c_2^{(21)}\left(2+\frac{p_3}{6s_1-4}\right),\\
a_{22}^{(2)} &= \frac{5+l_1c_1^{(21)}(1+6s_1)}{5l_1r_1}+c_2^{(21)}\left(-\frac{2}{r_2}+\frac{p_3}{4r_2-6r_2s_1}\right),
\end{aligned}
\tag{192}
$$

$$
\begin{aligned}
a_{11}^{(3)} &= \frac{11}{5}+\frac{6s_1}{5}+\frac{p_3}{6s_1-4}, \qquad a_{12}^{(3)} = -\frac{l_2\big(c_2^{(21)}+r_2c_2^{(22)}\big)(-8+p_3+12s_1)}{2r_2(3s_1-2)},\\
a_{21}^{(3)} &= \frac{2}{l_2}+\frac{c_1^{(21)}(1+6s_1)}{5}+c_2^{(21)}\left(2+\frac{p_3}{6s_1-4}\right),\\
a_{22}^{(3)} &= \frac{-5l_1r_1c_2^{(21)}(-8+p_3+12s_1)+2r_2(3s_1-2)\big(5+l_1\big(c_1^{(21)}+r_1c_1^{(22)}\big)(1+6s_1)\big)}{10l_1r_1r_2(3s_1-2)},
\end{aligned}
\tag{193}
$$

$$
\begin{aligned}
a_{11}^{(4)} &= \frac{6(1+s_1)}{5},\\
a_{12}^{(4)} &= \frac{1}{5r_1}\big(5+l_1\big(c_1^{(21)}+r_1c_1^{(22)}\big)(1+6s_1)\big)+\frac{p_3-l_2\big(c_2^{(21)}+r_2c_2^{(22)}\big)(-8+p_3+12s_1)}{2r_2(3s_1-2)},\\
a_{21}^{(4)} &= \frac{1}{l_1}+\frac{c_1^{(21)}(1+6s_1)}{5}, \qquad a_{22}^{(4)} = \frac{2}{l_2r_2}.
\end{aligned}
\tag{194}
$$

Note that not all constant matrices may be arbitrary diagonal matrices in equation (154). In fact, equation (186) means the following relation:

$$C^{(12)}R^{(2)}+L^{(2)}(C^{(21)}+C^{(22)}R^{(2)})+I_2=0. \tag{195}$$

Since all p_i are positive, the solution $V(t)$ (189) has no singularities unless $\sum_{m_1,m_2=1}^{2}|t_{m_1m_2}|\to\infty$. However, off-diagonal elements of V tend to infinity in some directions in the space of the parameters $t_{m_1m_2}$. Thus, V is not a bounded solution. Now we derive a simple example of the bounded soliton–kink solution. For this purpose we take

$$s_3=s_5=0, \qquad s_4=-1 \tag{196}$$

instead of equations (188). Then one obtains the following formula for V:

$$V(t)=\begin{bmatrix}\dfrac{f_{11}}{d+5\,\mathrm{e}^{\eta_{12}+\eta_{21}}} & \dfrac{f_{12}}{d\mathrm{e}^{-\eta_{12}}+5\,\mathrm{e}^{\eta_{21}}}\\ \dfrac{f_{21}}{d\mathrm{e}^{-\eta_{21}}+5\,\mathrm{e}^{\eta_{12}}} & \dfrac{f_{22}}{d+5\,\mathrm{e}^{\eta_{12}+\eta_{21}}}\end{bmatrix},$$

$$\begin{aligned}&f_{11}=\frac{6}{5}(3s_1-2)(6s_1+1)s_2, \qquad f_{12}=-3(6s_1+1)s_2,\\ &f_{21}=-2(3s_1-2)(3s_2+2), \qquad f_{22}=6(3s_1-2)(3s_2+2)s_2,\\ &d=-6(3s_1-2)s_2,\end{aligned} \tag{197}$$

where $\eta_{n_1n_2}$ are the linear functions of $t_{n_1n_2}$:

$$\begin{aligned}&\eta_{12}=\sum_{n_1,n_2=1}^{2}a_{n_1n_2}t_{n_1n_2}, \qquad \eta_{21}=\sum_{n_1,n_2=1}^{2}b_{n_1n_2}t_{n_1n_2},\\ &a_{11}=\frac{6}{5}(s_1+1), \qquad a_{12}=-\frac{1}{r_1r_2}\big(r_1\big(2+(3s_2+2)l_2\big(c_2^{(21)}+c_2^{(22)}r_2\big)\big)+r_2(a_{11}-2)\big),\\ &a_{21}=\frac{c_1^{(21)}}{5}(6s_1+1)+\frac{1}{l_1}-\frac{2}{l_2},\\ &a_{22}=\frac{1}{l_2r_1r_2}\big(r_1\big(2+(3s_2+2)l_2\big(c_2^{(21)}+c_2^{(22)}r_2\big)\big)-r_2(a_{21}l_2+2)\big),\\ &b_{11}=3s_2+1,\\ &b_{12}=-\frac{1}{5r_1r_2}\big(r_2\big(5+(6s_1+1)l_1\big(c_1^{(21)}+c_1^{(22)}r_1\big)\big)+5r_1(b_{11}-1)\big),\\ &b_{21}=c_2^{(21)}(3s_2+2)-\frac{1}{l_1}+\frac{2}{l_2},\\ &b_{22}=\frac{1}{5l_1r_1r_2}\big(r_2\big(5+(6s_1+1)l_1\big(c_1^{(21)}+c_1^{(22)}r_1\big)\big)-5r_1(b_{21}l_1+1)\big).\end{aligned} \tag{198}$$

If $d>0$, then the diagonal elements of matrix (197) are kinks, while off-diagonal elements tend to infinity in some directions in the space of the parameters $t_{m_1m_2}$. In order to obtain a bounded solution we require

$$\eta_{12}=a\eta_{21} \quad\Rightarrow\quad a_{ij}=ab_{ij}, \qquad a>0. \tag{199}$$

For the sake of simplicity, we solve equations (199) for the particular choice of the arbitrary parameters:

$$r_1=2, \qquad r_2=3, \qquad l_1=4, \qquad l_2=5, \qquad c_1^{(21)}=6. \tag{200}$$

One has

$$s_1=\frac{741+366c_2^{(22)}-12c_1^{(22)}\big(7+24c_2^{(22)}\big)}{8\big(-559+108c_2^{(22)}+36c_1^{(22)}\big(2+c_2^{(22)}\big)\big)}, \tag{201}$$

$$\begin{aligned}s_2=&\big(-\big(14\,521-181\,090c_2^{(22)}+43\,200\big(c_2^{(22)}\big)^2+144\big(c_1^{(22)}\big)^2\big(-1+20c_2^{(22)}\big)\\ &+40c_1^{(22)}\big(-20+9c_2^{(22)}+360\big(c_2^{(22)}\big)^2\big)\big)\\ &\times\big(40\big(-1+15c_2^{(22)}\big)\big(-559+108c_2^{(22)}+36c_1^{(22)}\big(2+c_2^{(22)}\big)\big)\big)^{-1},\end{aligned} \tag{202}$$

$$a=\frac{246-3690c_2^{(22)}}{-233+2160c_2^{(22)}+36c_1^{(22)}\big(-1+20c_2^{(22)}\big)}, \tag{203}$$

$$c_2^{(21)} = \frac{25 - 36\big(3 + c_1^{(22)}\big)c_2^{(22)}}{-89 + 12c_1^{(22)}}. \tag{204}$$

In addition, one has to provide positivity of $a = p_1 > 0$ in equation (197) and positivity of $d = p_2 > 0$. This requirement yields constraint for $c_i^{(22)}$, $i = 1, 2$:

$$\begin{aligned}
c_1^{(22)} &= \big(1240p_1 - p_1^3(725 + 144p_2) - p_1^2(525 + 248p_2) \\
&\quad \pm p_1(1 + p_1)\sqrt{25(-2 + p_1)^2 - 20p_1p_2}\big)\big(48p_1^2(-5 + p_1(5 + p_2))\big)^{-1}, \\
c_2^{(22)} &= \big(-1200p_1 + 5p_1^2(-263 + 48p_2) + 5p_1^3(497 + 100p_2) \\
&\quad \pm 3p_1(1 + p_1)\sqrt{25(-2 + p_1)^2 - 20p_1p_2}\big)\big(60(10p_1 - 5p_1^3 - p_1^2(15 + 2p_2) \\
&\quad \pm p_1(1 + p_1)\sqrt{25(-2 + p_1)^2 - 20p_1p_2})\big)^{-1}.
\end{aligned} \tag{205}$$

In particular, if $p_1 = 1/2$ and $p_2 = 1$, one has

$$\begin{aligned}
&s_1 = \frac{1 \pm \sqrt{185}}{24}, \qquad s_2 = \frac{15 \pm \sqrt{185}}{30}, \qquad c_2^{(21)} = \frac{3(-145 \pm 123\sqrt{185})}{2080}, \\
&c_1^{(22)} = \frac{-2545 \mp 3\sqrt{185}}{192}, \qquad c_2^{(22)} = \frac{70 \mp 93\sqrt{185}}{780}.
\end{aligned} \tag{206}$$

Now the expression for V, equation (197), reads

$$V(t) = -\begin{bmatrix} \dfrac{5 \pm \sqrt{185}}{20(1 + 5\,\mathrm{e}^{(a+1)\eta_{21}})} & \dfrac{(13 \pm \sqrt{185})}{2(\mathrm{e}^{-a\eta_{21}} + 5\,\mathrm{e}^{\eta_{21}})} \\ \dfrac{(-17 \pm \sqrt{185})}{2(\mathrm{e}^{-\eta_{21}} + 5\,\mathrm{e}^{a\eta_{21}})} & \dfrac{(35 \pm \sqrt{185})}{10(1 + 5\,\mathrm{e}^{(a+1)\eta_{21}})} \end{bmatrix}, \qquad \eta_{21} = \sum_{n_1,n_2=1}^{2} b_{n_1n_2}t_{n_1n_2}. \tag{207}$$

The appropriate expressions for $b_{n_1n_2}$ are as follows:

$$\begin{aligned}
b_{11} &= \frac{25 \pm \sqrt{185}}{10}, \qquad & b_{12} &= \frac{295 \pm 61\sqrt{185}}{30}, \\
b_{21} &= \frac{3(9 \pm 2\sqrt{185})}{10}, \qquad & b_{22} &= \frac{-214 \mp 43\sqrt{185}}{60}.
\end{aligned} \tag{208}$$

We see that diagonal elements of V represent kinks, while off-diagonal elements represent non-symmetrically shaped solitons.

Note that solution (207) causes some restrictions on the coefficients of equation (154), which are equations (204) and (205).

4. Conclusions

We represent a new algorithm allowing one to construct a rich variety of particular solutions to a new class of nonlinear PDEs of any order in any dimensions. These equations can be considered as multidimensional generalizations of well-known C- and S-integrable equations. We show that the solution space may be rich enough to provide complete integrability of some of these equations. However, the problem of complete integrability requires further study.

The suggested algorithm allows evident generalizations. For instance, let us generalize constraint (49) as follows:

$$\sum_{m,n=1}^{D} A^{(m)} * \tilde{A}^{(n)} B^{(mnp)} = \sum_{m=1}^{D} \tilde{A}^{(m)} P^{(mp)}, \qquad p = 1, \ldots, D, \quad j = 1, \ldots, m_0, \tag{209}$$

where $B^{(mnp)}$ and $P^{(mn)}$ are some constant $n_0 \times n_0$ matrices. Then one can show that the $n_0 \times n_0$ matrix functions $V^{(m)}(t)$ are solutions to the following system of nonlinear PDEs:

$$\sum_{m,n=1}^{D} \left[\left(V^{(n)}_{t_m} + (V^{(m)}V^{(n)} + V^{(m)}\mathcal{A}^{(n)})\right)B^{(mnp)}\right] + \sum_{m=1}^{D} V^{(m)}P^{(mp)} = 0,$$
$$p = 1, \ldots, D. \tag{210}$$

Remark that the physical application of some of the derived PDEs is obvious. For instance, the multidimensional *N*-wave equation (8) appears in multiple-scale analysis of any physical dispersion system. However, physical applications must be considered in more details.

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